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Environmental Regulation, Firm Heterogeneity and Macroeconomic Volatility

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Abstract

We build up an Heterogeneous Agent New Keynesian (HANK) model incorporating environmental regulation and heterogeneous firms in order to investigate the interrelationships between firm heterogeneity, environmental policy, and macroeconomic volatility. The findings are as follows: First, firm heterogeneity has relevant implications for macroeconomic volatility, regardless of the type of shock being analysed. Second, a cost-reducing technical change in abatement entails strong distributional changes, resulting in an aggregate efficiency gain. Third, a carbon pricing shock causes the aggregate reaction to the stricter environmental policy to be nonlinear, amplifying the macroeconomic response than the analogous Representative agent counterpart. The key micro-parameters are estimated consistently with data of regulated firms under the EU Emissions Trading System (EU ETS).

Keywords: HANK, E-DSGE, Environmental policy, Macroeconomic dynamics

JEL classification: E32, E62, F41, F44

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1. Introduction

The relationship between business cycle and environmental policies has been widely analysed in the context of Dynamic Stochastic General Equilibrium models, basically integrating environmental variables within this common workhouse of contemporary macroeconomics.[1] Starting from the early works by Bartz and Kelly (2008), Fischer and Springborn (2011), Heutel (2012) and Angelopoulos et al. (2013) this strand of literature has evolved along many directions. Some contributions highlighted the role of market imperfections and nominal rigidities in altering the performance of environmental regulations (see Annicchiarico and Di Dio (2015)), some others the role of labour market frictions in affecting the interaction between business cycle and emissions volatility (see Gibson and Heutel (2020)), while a more recent literature points to credit market imperfections and unconventional monetary policy (see Diluiso et al. (2021) and Carattini et al. (2021)). See also Annicchiarico and Diluiso (2019) for an open-economy extension and Annicchiarico et al. (2018) for model with environmental policy and endogenous market structure.[2] However, only few contributions have analysed the role of heterogeneity in accounting for aggregate volatility in the presence of environmental policy.[3]

This paper provides a fresh look at the debate about the relationship between business cycle, environmental policy and macroeconomic volatility and in the presence of several sources of aggregate uncertainty and firm-level heterogeneity.

To this end, we extend a prototypical Real Business Cycle model along the lines of an emerging class of macroeconomic models, namely the Heterogeneous Agent New Keynesian (HANK) models,[4] which provide a sound framework for quantitative analysis able to capture the interplay between aggregate outcomes and firm distributions. Although the Representative agent models still remain the benchmark methodology for analysing aggregate fluctuations, the HANK models provide a more accurate representation of firm behavior subject to the environmental regime but explicitly accounting for the interaction

[1] Conventionally, basic DSGE models augmented to include some aspects of the environment have been named E-DSGE models (see Khan et al. (2019)).

[2] For a recent comprehensive review of the literature related to business cycle and environmental policy see Annicchiarico et al. (2022).

[3] We will discuss this relevant literature later on in this paragraph.

[4] While a great deal of the HANK literature focuses on households heterogeneity, in this paper we only deal with firm heterogeneity. This last stand of literature mainly debates about the role of firm-level non-convex adjustment costs and aggregate investment (see Bachmann et al. (2013), Khan and Thomas (2008) and more recently Winberry (2021)), while more recently some contributions studied the role of financial frictions and firm heterogeneity in determining the transmission of monetary policy (see Ottonello and Winberry (2020) and the role of financial frictions in determining the allocation of investment and innovation Ottonello and Winberry (2024).

between firm cross-sectional distribution and aggregate uncertainty.

The model we construct departs from the standard E-DSGE model as it includes two specific features.[5]

First, we introduce an idiosyncratic productivity shock to the production function. In this way, the traditional aggregate productivity shock coexists with firm-specific uncertainty. As a result, the macroeconomic response to aggregate business cycle shocks takes into account the heterogeneity caused by firm-level differentiated productivity. Furthermore, because the economy is composed of many heterogeneous firms, we can appropriately examine the role of the environmental regulation on abatement decisions when the production structure is heterogeneous and the source of the business cycle is different.

Second, firms undertake extensive-margin investment decisions. This means that variations in the number of firms embarking on new investment projects (the extensive margin) impact the aggregate investment dynamics more than changes in the size of current investment projects (the intensive margin, as in the prototypical E-DSGE model). In the model, given the aggregate shock, a share of firms makes investment decisions taking into account a fixed capital cost. This cost gives rise to the extensive margin decision of whether to invest or not.

In this way, we include in the model the empirically documented investment lumpiness,[6] that is the fact that investments at the micro level appear to be composed of isolated spikes more than by smoothed variations. This feature is thus important to reconcile the dynamics of macro-variables with the behaviour of investments at the micro level. Besides, to capture this feature in the model's parametrization coupled with the presence of environmental policy, we estimate some key parameters consistently with data of regulated firms under the EU Emissions Trading System (EU ETS) (see Section 4.3 for more details).

Both these features lead to macroeconomic responses that are different from the traditional E-DSGE model. As a result of business cycle shocks, aggregate investments from heterogeneous firms are more sensitive to the current business cycle conditions and less sensitive to interest rate. Indeed, being now the investment decisions mostly based on the extensive margin, the elasticity of aggregate investment relative to the shocks is procyclical; this implies that in expansions, more firms are on the verge of making an extensive margin investment, so that any further shock produces more total investment than it would be in the Representative agent counterpart. On the contrary, when the economy is

[5] These properties, however, are shared by many models that assume heterogeneity on the supply side of the economy.

[6] See among others, Khan and Thomas (2008) and more recently Winberry (2021) for the US economy.

perturbed by recessionary shocks a higher mass of firms will reduce capital, resulting in a stronger negative variation of investments than in the standard E-DSGE model.

Given the state-dependency of investment decisions, interest rate movements are less relevant in aligning the demand components with household preferences and production. In other terms, the predicted relationship between interest rate on the one hand, and consumption, investment and abatement costs on the other hand, is weaker than the E-DSGE counterpart.

We illustrate this point with a series of shocks, highlighting the similarities and differences between the Heterogeneous framework (henceforth, Het) and the Representative agent version (henceforth, Rep).[7] In particular, the paper explores the transmission of shocks commonly considered in business cycle literature with a focus on the role played by firm heterogeneity in driving the dynamic responses of the economy under the Cap-and-Trade regime, where the aggregate level of emissions is set and the government auctions emission permits to the producers at the market price.

We thus allow for different sources of macroeconomic uncertainty. Specifically, we analyse the dynamic response of the economy to five shocks, namely: (i) a technology shock to the Total Factor Productivity (TFP), (ii) an emission-intensity shock, (iii) a saving shock, (iv) an shock to the marginal efficiency of investment, (v) a carbon pricing shock. The first two shocks concern the supply side of the economy while the third and the forth are related to the demand side. Carbon pricing is a policy shock that involves an exogenous change in the carbon price.

The model is calibrated on the European (EU27) economy. The aggregate stochastic processes of the model (degree of persistence and standard deviation) are estimated using Bayesian methods. By merging data from several sources, a sample of firms that are subject to the EU Emissions Trading System (EU-ETS) is used to estimate the micro-level parameters that influence firm-level investment decisions. In particular, given the economic parameters and the estimated aggregate shock, we calibrate the micro parameter of the model using the distributional data of regulated companies of this data set.[8]

So far, the interplay between firm heterogeneity, climate actions and economic uncertainty has not been investigated in this context, so that this type of extension has still

[7] The Rep model closely follows the structure of the model described in the next Section but it does not feature firm specific productivity. Also, investment decisions relies on the intensive margin. For a full description of the corresponding equilibrium conditions of the Representative model (Rep model) described in the next section, see Appendix B.

[8] Approximately 45000 firm-year observations make up the final database, which contains information on investment statistics for firms under the EU-ETS system from 2005 to 2021. See Section 4.3 for more details.

remained unexplored. To the best of our knowledge, this paper is the first contribution to study the transmission of business cycle shocks under climate regulation following the HANK approach in the presence of heterogeneous firms.

The closest predecessors of our paper analyse the distributional impact of climate actions from a different angle. Some recent examples include Jondeau et al. (2022), Coria and Kyriakopoulou (2018), Anouliès (2017) and Dissou and Karnizova (2016).

Jondeau et al. (2022) develop and estimate an E-DSGE model for the world economy in which the adoption of new abatement technologies and the creation of startups are endogenous, following the approach by Bilbiie et al. (2012). They find that public subsidies, financed by a carbon tax, are an efficient instrument to promote firm entry into the abatement goods sector and in mitigating the transition risk associated with a long-term climate-neutral objectives. The analysis takes into account several important points about climate actions and firm heterogeneity by means of entrants and incumbents operating in the abatement goods sector. Nonetheless, the model does not account for the uncertainty about future economic and climate conditions.

Coria and Kyriakopoulou (2018) study the effects of three environmental policies such as emission taxes, uniform emission standards, and performance standards on the size distribution of firms. Firms produce a homogeneous good using energy and labor and differ in terms of energy efficiency. The model is theoretical and static, and relies on some simplifying assumptions about the energy efficiency distribution. The find that emission standards allow for regulatory asymmetries favoring small firms, while emission taxes and performance standards reduce to a lower extent profits of larger firms although they do modify the optimal scale of firms.

Anouliès (2017) analyses the economic and environmental effects of the European Union Emissions Trading Scheme (EU ETS) in a general equilibrium model with heterogeneous firms and monopolistic competition in the spirit of Melitz (2003). Firms feature heterogeneous marginal labor productivity and generate pollution which is regulated by a cap-and-trade program imposing a cap on emissions. She finds that the cap on emissions has no effect on firms' profits, or decisions to enter or exit the market. Firm heterogeneity also magnifies the economic effects of changes in the initial allocation of allowances, reallocating resources among firms towards the most productive ones which have an impact on firms' entry and exit decisions.

Dissou and Karnizova (2016) construct a multi-sector business cycle model to scrutinise the economic implications of reducing emissions with two alternative regimes, Cap-and-Trade or Carbon Taxes. The model also includes multiple sources of macroeconomic uncertainty and is calibrated for the US economy. Relative to previous studies, the multi-sector analysis highlights the importance of the origin of the shocks to rank of the two available instruments. Shocks with a direct impact on the level of emissions play a critical role in ranking the cap and the tax while the ranking of the cap and the tax using the volatility and welfare criteria does not coincide.

We derive three key insights that can be summarised as follows.

First, firm heterogeneity has relevant implications for macroeconomic volatility in the presence of environmental regulation, regardless of the source of the business cycle variation. Although two models show some qualitative similarities, supply as well as demand side shocks show quantitative discrepancies coming from different propagation mechanisms at play.

Second, a cost-reducing technical change in abatement entails some strong distributional changes in the Het version, leading to an aggregate efficiency gain. This implies that, as a result of this shock, more efficient firms with larger capital will operate in the market. On the opposite, being this mechanism is absent in the Rep version, abatement costs follow a positive variation in this version, since more effort in abating is needed to comply with the emission cap.

Third, a carbon pricing shock is responsible for stronger adjustment of investment and consumption due to a distributional shift to less efficient abatement than in the Rep version. Indeed, the dynamics of the distribution of capital stocks and productivity across productive plants cause the aggregate reaction to the strict environmental policy to be nonlinear, implying a different macroeconomic adjustment than the analogous representative agent counterpart. The paper is structured as follows. Section 2 presents the model, while Section 3 shows the solution of the optimization problem. Section 4 reports the model solution and the econometric approach followed for estimating the aggregate stochastic processes and the micro parameters relying on investment decisions of firms subject to the EU Emissions Trading System (EU-ETS). Section 5 reports the results. Section 6 concludes.

2. Model Setup

We consider a prototypical RBC model extended to include environmental regulation, firm-specific productivity and extensive-margin investment decisions.[9] The model also allows for environmental externality. We first start with the description of the model setup; then, we delve into the optimization problem of households and firms. We conclude

[9]The model is basically an heterogeneous variant of Bartz and Kelly (2008). See Heutel (2012) for a complete exposition of the basic RBC model commonly used in this literature.

with the definition of the related equilibrium.

2.1. Model Description

The generic producer i has access to the following technology $q_{i,t}$:

$$q_{i,t} = \Delta_t e^{A_t} e^{\epsilon_{i,t}} n_{i,t}^{\alpha} k_{i,t}^{\beta}, \tag{1}$$

where Δ_t is the negative environmental externality, $n_{i,t}$ is the firm-specific labour factor, $k_{i,t}$ is the firm-specific capital stock and the parameters α and β satisfy $\alpha + \beta < 1$.[10] The environmental externality[11] Δ_t is defined as follows :

$$\Delta_t = e^{-\chi(M_t - \tilde{M})}, \tag{2}$$

where M_t is the aggregate stock of firm pollution and $\tilde{M}$ is the pre-industrial atmospheric concentration of Greenhouse Gases (GHG) and χ is a positive scale parameter measuring the intensity of the negative externality on productivity or analogously the fraction of production lost for each extra unit of pollutants. The equation describes how the greenhouse gas concentration in the atmosphere translates into the economic damage.[12]

Furthermore, A_t is the aggregate TFP shock common to all firms following an AR(1) process:

$$A_t = \rho_A A_{t-1} + \epsilon_{A,t}, \quad \epsilon_{A,t} \sim N(0,1) \tag{3}$$

[10] Miao and Wang (2014) were the first to show the irrelevance of fixed costs for aggregate dynamics if constant returns to scale are present, without elucidating the economic mechanism. Koby and Wolf (2020) and Winberry (2021) report that, in a general class of heterogeneous firm models, the price elasticity of investment in the limit of constant return to scale diverges. Therefore, if the returns to scale are constant or nearly constant, small but procyclical shocks in the real prices bring aggregate investment in line with the representative household's desired path of consumption, regardless of the existence of fixed costs.

[11] Damages from climate change include, among other factors, loss of life, deterioration in the quality of life, and depreciation of the capital stock. These damages should also include any resources used to prevent disasters and, more generally, to lessen the impact of climate change on humans and human activity. See Golosov et al. (2014) for further details.

[12] This type of formalization is well documented in the literature. The DICE/RICE family of models introduces an exponential version of the well-known Nordhaus damage function (see, for example, Nordhaus (2018)). According to Nordhaus, there is a link between rising global temperatures and economic loss. Nordhaus, on the other hand, explicitly models damages in two phases, the first of which maps carbon concentration onto temperature and the second mapping temperature to damages. Golosov et al. (2014) present a function directly mapping temperature to damages. connecting the stock of carbon dioxide to economic damages. As with the RICE and DICE, the damage effects are multiplicative. The exponential specification is thus a fair approximation of Nordhaus requirements, as discussed by the authors.

The aggregate stock of emissions[13] evolves as follows:

$$M_t = Z_t + Z_t^{RoW} + (1 - \delta_M)M_{t-1}, \tag{4}$$

where $\delta_M \in (0, 1)$ measures the natural rate at which the atmosphere recovers, Z_t is for the domestic aggregate emissions, namely the sum of the emissions of every domestic firm and Z_t^{RoW} is an exogenous process measuring the Rest-Of-the-World emissions. In what follows we will assume that Z_t^{RoW} is constant.

Furthermore, the idiosyncratic productivity shock $\epsilon_{i,t}$ is the first source of heterogeneity in the model. The shocks are independent across firms and follow an AR(1) process:

$$\epsilon_{i,t} = \rho_\epsilon \epsilon_{i,t-1} + \eta_t, \quad \eta_t \sim N(0, 1). \tag{5}$$

Investment behavior is induced by the second source of heterogeneity in the model. In this sense, there is a mass of firms which make investment decisions but subject to idiosyncratic adjustment cost, while a fraction of other firms will not change the capital.

The idiosyncratic fixed adjustment cost $\psi_{i,t} \sim U(0, \bar{\psi})$ that the firm has to pay in order to change its next period capital $k_{i,t+1}$ is expressed in unit of labor.

More formally, the law of motion of capital is:

$$k_{i,t+1} = \begin{cases} (1-\delta_k)\, k_{i,t} + e^{\Lambda_{q,t}} i_{i,t} & \text{if} \quad i_{i,t} \neq 0 \quad \text{paying} \quad \psi_{i,t} w_t \\ \\ (1-\delta_k)\, k_{i,t} & \text{if} \quad i_{i,t} = 0 \quad \text{paying} \quad 0 \end{cases} \tag{6}$$

where δ_K is the capital depreciation and $\Lambda_{q,t}$ is the aggregate shock to to the marginal efficiency of investment following an AR(1) process as follows:

$$\Lambda_{q,t} = \rho_{\Lambda_q} \Lambda_{q,t-1} + \epsilon_{\Lambda_q,t}, \quad \epsilon_{\Lambda_q,t} \sim N(0, 1). \tag{7}$$

The investment shock $\Lambda_{q,t}$ affects the extent to which resources are allocated to investment thus changing the effective quantity of capital available for production.

Every firm is subject to the environmental policy and has to bear some costs in order to comply with the emission target. Following Bartz and Kelly (2008) we assume the cost

[13] It is worth noticing that the presence of this global stock of emissions as externality is quantitatively irrelevant for the single firm, while its dynamics might change as result of the environmental regime at play.

of emissions abatement has the following functional form:

$$Cost(u_{i,t}) = \left[1 - (1 - u_{i,t})^{\zeta}\right] q_{i,t}, \quad 0 < \zeta < 1, \tag{8}$$

with $0 \leq u_{i,t} \leq 1$ be the abatement (in terms of output). The production function, net of abatement costs, is thus:

$$y_{i,t} = (1 - u_{i,t})^{\zeta} q_{i,t}. \tag{9}$$

Emissions at firm level, $z_{i,t}$, are assumed to be proportional to output, net of the abatement effort $u_{i,t}$:

$$z_{i,t} = \frac{1 - u_{i,t}}{\phi e^{\Omega_t}} q_{i,t}, \tag{10}$$

where $\frac{1-u_{i,t}}{\phi e^{\Omega_t}}$ is the emission intensity of output and Ω_t is the corresponding aggregate emission-intensity shock common to all firms following an AR(1) process:

$$\Omega_t = \rho_{\Omega} \Omega_{t-1} + \epsilon_{\Omega,t}, \quad \epsilon_{\Omega,t} \sim N(0,1). \tag{11}$$

This shock is assumed to capture technological change in abatement and compositional changes in output. Given that the total cost of emissions equals $q_{i,t} - y_{i,t}$, which is decreasing in Ω, it can be viewed as cost-reducing technical change in abatement (high Ω_t induces a (relative) low abatement cost).

Combining (9) with (10) delivers:

$$y_{i,t} = \left(\phi e^{\Omega_t} z_{i,t}\right)^{\zeta} \left(\Delta_t e^{A_t} e^{\epsilon_{i,t}}\right)^{1-\zeta} n_{i,t}^{\nu} k_{i,t}^{\theta}. \tag{12}$$

where $\nu = (1-\zeta)\alpha$ and $\theta = (1-\zeta)\beta$. In this formalization $y_{i,t}$ is the output net of abatement costs, the parameter θ is the elasticity of output with respect to capital, and ν is the elasticity of output with respect to labor and ζ can thus be interpreted as the emission share.

Concerning the demand side, the economy is populated by a *continuum* of length one of infinitely lived households with preferences represented by:

$$E_t \sum_{t=0}^{\infty} \beta^t \left(e^{\Lambda_{c,t}} \frac{C_t^{1-\gamma}}{1-\gamma} - \chi \frac{N_t^{1+\varphi}}{1+\varphi} \right), \tag{13}$$

where E denotes the expectation operator, β is the discount factor, C_t is the aggregate consumption, while N_t denotes aggregate hours of work. The parameter γ is the coefficient of risk aversion, χ is the weight of the disutility of working, φ is the inverse of Frisch elasticity.

Furthermore, $\Lambda_{c,t}$ is the aggregate shock to consumption following an AR(1) process as follows:

$$\Lambda_{c,t} = \rho_{\Lambda_c}\Lambda_{c,t-1} + \epsilon_{\Lambda_c,t}, \quad \epsilon_{\Lambda_c,t} \sim N(0,1) \tag{14}$$

The household owns all firms in the economy and markets are complete. Hence each household faces a flow budget constraint of the form:

$$C_t + B_{t+1} - B_t \leq w_t N_t + \Pi_t + T_t + r_{t-1}B_t, \tag{15}$$

where Π_t are the aggregate profits, w_t is the real wage and T_t represents the lump-sum component of income including transfers from the public sector; B_t is the bond holdings; $r_{t-1}B_t$ is the interest income on the existing bond held.[14]

3. Optimization

In this Section we first present the relevant equations related to the optimization problem of households (section 3.1) and firms (section 3.2). We then define the environmental policy (section 3.3) and the Recursive Competitive Equilibrium (section 3.4).

We also change the mathematical notation of the variables in the optimization problems. Aggregate variables are now expressed as a function of the aggregate state $\boldsymbol{s}$, which includes the current draws of aggregate shocks as well as the current distribution of productivity shocks and capital across companies. Lower case is for a variable at the firm level.[15]

3.1. Household optimization

The representative household optimization problem boils down to the usual problem related to the choice of consumption, labour and bonds. The first-order conditions from the utility maximization problem are the following:

$$\lambda(\boldsymbol{s}) = e^{\Lambda_c} C(\boldsymbol{s})^{-\gamma} \tag{16}$$

[14] Note that this term can be negative implying that r is an interest cost of servicing debt. In other terms, $B_t > 0$ means that the household has a positive stock of savings; $B_t < 0$ means the household has a stock of debt.

[15] For the time convention we closely follow Winberry (2021).

$$w(\boldsymbol{s}) = \frac{\chi N(\boldsymbol{s})^{\varphi}}{\lambda(\boldsymbol{s})} \tag{17}$$

$$1 + r(\boldsymbol{s}) = \frac{\lambda(\boldsymbol{s})}{\beta\lambda(\boldsymbol{s'})} \tag{18}$$

where $\lambda(\boldsymbol{s})$ is the Lagrange multiplier associated to the budget constraint (15) and $r(\boldsymbol{s})$ is the risk-free real interest rate, taken as given in the optimization problem.

Equation (16) defines the marginal propensity of consumption; equation (17) equates the marginal rate of substitution between leisure and consumption ($\frac{\chi N^{\varphi}(\boldsymbol{s})}{\lambda(\boldsymbol{s})}$) to the relative price of leisure, that is $w(\boldsymbol{s})$; equation (18) equates the marginal rate of substitution between consumption today and tomorrow ($\frac{\lambda(\boldsymbol{s})}{\beta\lambda(\boldsymbol{s'})}$) to the relative price of consumption today (that is, $1 + r(\boldsymbol{s})$).

Markets are complete with respect to the aggregate risk, so that the price of output used by firms is equal to the household's marginal utility of consumption $\lambda(\boldsymbol{s})$.

3.2. Firm optimization

The demand for labor immediately follows by solving the simple intra-temporal optimization problem of generic producer:

$$w(\boldsymbol{s}) = \nu \left(\phi e^{\Omega(\boldsymbol{s})} z\right)^{\zeta} \left(\Delta(\boldsymbol{s}) e^{A(\boldsymbol{s})} e^{\epsilon}\right)^{1-\zeta} n^{\nu-1} k^{\theta} \tag{19}$$

Firms must pay an emission price (an emission tax) $p_z(\boldsymbol{s})$, therefore at the optimum the following condition must hold:

$$p_z(\boldsymbol{s}) = \zeta\, (z)^{\zeta-1} \, (\phi e^{\Omega(\boldsymbol{s})})^{\zeta} \left(\Delta_t e^{A(\boldsymbol{s})} e^{\epsilon}\right)^{1-\zeta} n^{\nu} k^{\theta} \tag{20}$$

where the firm equates the marginal product of abatement to its marginal cost.

The firm will choose current hours n, emissions z and next period capital k', given current productivity shock ϵ, current capital stock k, the current aggregate state $\boldsymbol{s}$ of the economy according to the following value function:

$$\begin{aligned} v(\epsilon, k, \psi; \boldsymbol{s}) \;&=\; \max_{n,z} \left\{ \left(\phi e^{\Omega} z\right)^{\zeta} (e^{A} e^{\epsilon} \Delta(\boldsymbol{s}))^{1-\zeta} n^{\nu} k^{\theta} - w(\boldsymbol{s}) n - p_z z \right\} \\ &+\; \max \left\{ v_{\{i \neq 0\}}(\epsilon, k; \boldsymbol{s}) - \psi w(\boldsymbol{s}), v_{\{i=0\}}(\epsilon, k; \boldsymbol{s}) \right\} \end{aligned}$$

The optimization problem for labour and emissions is static and could be solved before tackling the dynamic programming problem for capital. In particular their solution is:

$$n(\epsilon,k;\boldsymbol{s}) = \left(\frac{p_z(\boldsymbol{s})}{\zeta\phi e^{\Omega}}\right)^{\frac{\zeta}{\nu+\zeta-1}} \left(\frac{\nu e^{\epsilon}e^{A}\Delta(\boldsymbol{s})}{w(\boldsymbol{s})}\right)^{\frac{\zeta-1}{\nu+\zeta-1}} k^{-\frac{\theta}{\nu+\zeta-1}} \tag{21}$$

$$z(\epsilon,k;\boldsymbol{s}) = n(\epsilon,k;\boldsymbol{s})\frac{w(\boldsymbol{s})}{p_z(\boldsymbol{s})}\frac{\zeta}{\nu} \tag{22}$$

In the optimization problem for the capital we distinguish between the value function conditional on investing [16], that is:

$$v_{\{i\neq 0\}}(\epsilon,k;\boldsymbol{s}) = \max_{k'}\left\{-(k'-(1-\delta)k) + E\left[\Lambda(a';\boldsymbol{s}))\hat{v}(\epsilon',k',\boldsymbol{s'})|\epsilon,k;\boldsymbol{s}\right]\right\}, \tag{23}$$

where $\Lambda(a';\boldsymbol{s}) = \frac{\beta\lambda(\boldsymbol{s'})}{\lambda(\boldsymbol{s})}$ is the discount factor, while the value function conditional on not investing:

$$v_{\{i=0\}}(\epsilon,k;\boldsymbol{s}) = E\left[\Lambda(a';\boldsymbol{s})\hat{v}(\epsilon',k',\boldsymbol{s'})|\epsilon,k;\boldsymbol{s}\right]. \tag{24}$$

In the above expressions $\boldsymbol{s}' = \boldsymbol{s}'(\boldsymbol{s})$ is the law of motion of the aggregate state and:

$$\hat{v}(\epsilon',k';\boldsymbol{s'}) = E\left[v(\epsilon',k',\psi',\boldsymbol{s'})|\epsilon',k';\boldsymbol{s}'\right] = \int_0^{\bar{\psi}} v(\epsilon',k',\psi',\boldsymbol{s}')\frac{1}{\bar{\psi}}d\psi' \tag{25}$$

is the conditional expected value of the value function. Given the assumption of uniformity on the distribution of the adjustment cost, integration could be carried out analytically.

In particular the firm will invest if:

$$v_{\{i\neq 0\}}(\epsilon,k;\boldsymbol{s}) - \psi\lambda(\boldsymbol{s})w(\boldsymbol{s}) \geq v_{\{i=0\}}(\epsilon,k;\boldsymbol{s}) \tag{26}$$

Since the firm finds it advantageous to pay the fixed cost only on an occasional basis, it results in the irregular (lumpy) investment patterns consistently to the observed behavior in the micro data.

Using an approximation for $\hat{v}$ and numerical quadrature for the expectations[17] we can solve iteratively for $k_{i\neq 0}$ and $\hat{v}$.

[16] For the sake of simplicity in the optimization problem we omit the aggregate shock on investment $\Lambda_{q,t}$ as described in (7). For more technical details about the optimization problems see Appendix C

[17] Details of the type of approximation and quadrature used can be found in Winberry (2018).

3.3. *Environmental Policy*

We consider the Cap-and-Trade as the environmental policy regime.[18] Under this regulation the aggregate level of emissions that can be released is set so that:

$$\int z\left(\epsilon,k;\boldsymbol{s}\right)g(\epsilon,k)dkd\epsilon=\overline{Z} \tag{27}$$

where $\overline{Z}$ is the emission cap, $g(\epsilon,k)$ is the probability distribution of firms over their individual states. The government sells emission permits to the producers at the market price $p_z(\boldsymbol{s})$. Abstracting from the presence of public debt, the budget constraint of the public sector reads as:

$$T(\boldsymbol{s})=p_z(\boldsymbol{s})\overline{Z} \tag{28}$$

where $p_z(\boldsymbol{s})\overline{Z}$ are the revenues collected from firms through the government sale of emission permits for given the cap. The revenues from the environmental policy are thus assumed to be rebated back to households as lump-sum transfers $T(\boldsymbol{s})$.[19] In setting the optimal level of emissions each firm must strike a balance between the additional cost related to a major abatement effort and the price to pay for each additional unit of emissions. In this case the price is endogenous and is determined by the demand of emission permits in the market, being the supply fully elastic.

3.4. *Definition of Equilibrium*

Once the above optimization problem is solved we can define the Recursive Competitive Equilibrium as follows:

Definition 1. *Define the aggregate state* $\boldsymbol{s}=(A,\Omega,\Lambda_c,g)$*, a Recursive Competitive Equilibrium for this model is a set of functions* $\hat{v}\left(\epsilon,k;\boldsymbol{s}\right)$, $z\left(\epsilon,k;\boldsymbol{s}\right)$, $n\left(\epsilon,k;\boldsymbol{s}\right)$, $k_{i\neq 0}\left(\epsilon,k;\boldsymbol{s}\right)$, $\hat{\psi}\left(\epsilon,k;\boldsymbol{s}\right)$, $p_z\left(\boldsymbol{s}\right)$, $w\left(\boldsymbol{s}\right)$ *such that:*

(i) (Firm optimization) Taking $p_z\left(\boldsymbol{s}\right),\lambda(\boldsymbol{s}),w(\boldsymbol{s})$ *and* $\boldsymbol{s}'\left(A',\boldsymbol{s}\right)$ *as given,* $z\left(\epsilon,k;\boldsymbol{s}\right)$ *is given by* (22), $n\left(\epsilon,k;\boldsymbol{s}\right)$ *is given by* (21) , $\hat{v}\left(\epsilon,k;\boldsymbol{s}\right), k_{i\neq 0}\left(\epsilon,k;\boldsymbol{s}\right), \hat{\psi}\left(\epsilon,k;\boldsymbol{s}\right)$ *is the solution of* (23) *and* (24).

(ii) (Implications of household optimization) Households will solve their optimization plan taking in $C(\boldsymbol{s})$ *taking* $w(\boldsymbol{s})$, $\Pi(\boldsymbol{s})$ *and* $T(\boldsymbol{s})$ *as given.*

[18]The model is potentially able to incorporate also a carbon tax. In the tax policy regime the government levies taxes on emissions at a constant (real) rate τ, implying that $p_z(\boldsymbol{s})=\tau$. As a result, the marginal cost of polluting is constant and so also the abatement.

[19]We rule out the possibility of grandfathering for the sake of simplicity.

(iii) (Market clearing in the good market) For all $\boldsymbol{s}$*:*

$$C(\boldsymbol{s}) \quad = \quad \int \left(\phi e^{\Omega} z(\epsilon, k; \boldsymbol{s})\right)^{\zeta} (e^{A} e^{\epsilon} \Delta(\boldsymbol{s}))^{1-\zeta} n(\epsilon, k; \boldsymbol{s})^{\nu} k^{\theta} g\left(\epsilon, k\right) d\epsilon dk - I\left(\boldsymbol{s}\right) \quad (29)$$

where $I\left(\boldsymbol{s}\right) = \int p_{\psi}(\epsilon, k; \boldsymbol{s}) \left[k_{\{i\neq 0\}}(\epsilon, k; \boldsymbol{s}) - (1-\delta_k)\, k\right] g\left(\epsilon, k\right) d\epsilon dk$

(iv) (Public Budget Implications) In the public sector the following relationship holds:

$$T(\boldsymbol{s}) = p_z \overline{Z} \quad (30)$$

where $\overline{Z} = \int z\left(\epsilon, k; \boldsymbol{s}\right) g\left(\epsilon, k\right) d\epsilon dk$

(v) (Law of Motion for Distribution)

$$\begin{aligned} g'(\epsilon', k') = \int \mathbf{1}\{\rho_\epsilon \epsilon + \eta = \epsilon'\} \left\{p_{\psi}(\epsilon, k; \boldsymbol{s}) \mathbf{1}\{k' = k_{i\neq 0}(\epsilon, k; \boldsymbol{s})\} + \right. \\ \left.\left(1 - p_{\psi}(\epsilon, k; \boldsymbol{s})\right) \mathbf{1}\{k' = (1-\delta_k)\, k\}\right\} p(\epsilon_\epsilon) g(\epsilon, k) d\epsilon_\epsilon d\epsilon dk \end{aligned} \quad (31)$$

where $p(\epsilon_\epsilon)$ *is the p.d.f of idiosyncratic productivity shock.*

4. Model solution and econometric approach

The computational method relies on Winberry (2018). With this methodology is feasible to approximate the distribution of firms (typically an infinite-dimensional object) using a flexible parametric family, reducing its dimensionality to a finite set of endogenous parameters, and solve for the dynamics of these endogenous parameters by perturbation. More broadly, the method employed in this research is related to a huge body of work that approximates the distribution with a small number of moments (essentially following the pioneering work of Krusell and Smith (1998)).

The model parametrization follows three steps. In the first step, a subset of parameters (the deep macro parameters) is calibrated at quarterly frequency to match long-run data properties for Europe (EU27) (see Section 4.1). In the second step, given the values of the above parameters the aggregate stochastic processes of the model (degree of persistence and standard deviation) are estimated using Bayesian methods. See Section 4.2. In the third step, we fix a set of parameters to match micro-level empirical targets of investments for firms under the EU ETS system (see Section 4.3). Section 4.4 presents the statistics of the model validation.

4.1. Calibration of the deep macro parameters

Table 1 provides an overview of the calibrated parameters. The model frequency is quarterly and it is solved by linearising it around its deterministic steady-state. The discount factor is set to 0.995 implying a steady-state annualised real interest rate by around 4% (EU27 data). Following Smets and Wouters (2003) we set the intertemporal elasticity of substituion σ to 1.35 while the inverse of the Frisch elaticity of labour supply is set to 1 that is an intermediate value within the range of micro and macro estimates. The depreciation rate, δ, is set equal to 0.025 while we set the labour share α to 0.64 according to the EU data (Eurostat data 2021) and the capital share to 0.21 according to Winberry (2021).

Concerning the environmental part of the model, we refer to previous E-DSGE models and Integrated Assessment Models for climate change, in order to obtain plausible values for environmental parameters. The calibration strategy delivers implicit values for the pollution stock in model units, emission intensity and the price of emission permits. When the economy is hit by shocks, however, the price of emission permits will change reflecting the changes in the market conditions. We set the pollution decay as in Heutel (2012); the steady state of carbon dioxide in the atmosphere, M, is set to 891.34 in gigatons of carbon, consistently with the value set for year 2020 in the base scenario of the DICE model.[20] Regarding the negative externality on production, we calibrate χ on the basis of the total damage for year 2020, measured as fraction of output, that amounts to 0.002438.[21] Estimating that the pre-industrial atmospheric CO_2 concentration ($\tilde{M}$) represents 75% of the total pollution stock, we obtain a value for the intensity of negative externality on output χ. The calibration of emission share ζ is computed using data related to environmental taxes as a percentage of GDP.[22] In particular, we set ζ=0.024 in order to match the observed proportion of environmental tax revenues in Gross Domestic Product (GDP) for the EU27 over 2007-2019.[23] Eventually, we calibrate the carbon pricing $P_{Z,t}$ to match the carbon dioxide emissions, expressed as thousand tonnes per Value Added, recorded by the European Union (27 countries) in 2019. Emissions from the rest of the world

[20]For details on the DICE model, see Nordhaus (2017, 2018). The DICE scenario does not include the COVID crisis and its effects on economic and environmental variables

[21]In alternative, Carattini et al. (2021) consider the mean value of the carbon stock over the first 250 years of the simulation in the DICE business-as-usual scenario. This implies a higher damage than that considered in the current calibration.

[22]Following Bartz and Kelly (2008) there are three alternatives to set the emission share ζ. We calibrate it as the share of income spent in emission taxes. This has the advantage to rely on higher quality data relative to the other options.

[23]Eurostat data 2021. There are four tax categories that make up environmental taxes: energy taxes, pollution taxes, resource taxes (excluding taxes on oil and gas extraction) and transport taxes. The environmental tax base is a physical unit of something that has a proven specific negative impact on the environment.

are assumed to be constant over time. This parametrization leads to a semi-elasticity of aggregate investment with respect to the real interest rate by around -84.56, somehow higher than in Winberry (2021) but much lower than in Khan and Thomas (2008) calibration.

Table 1: Calibrated Parameters

Parameter	Value	Description
β	0.995	Quarterly discount factor
γ	1.35	Risk aversion coefficient
φ	1.00	Inverse Frisch elasticity
α	0.64	Labor share
β	0.21	Capital share (Het)
δ	0.025	Quarterly capital depreciation rate
χ	1.09E-05	Intensity of negative externality on output
δ_M	0.0021	Emission decay rate
ζ	0.0240	Emission share

4.2. Estimation of the aggregate stochastic processes

The persistence and the standard deviations of the aggregate shocks are estimated using Bayesian methods. As in Bayesian practice, the likelihood function (evaluated by implementing the Kalman Filter) and the prior distribution of the parameters are combined to calculate the posterior distribution.[24] The posterior Kernel is then simulated numerically using the slice sampler algorithm as proposed by Planas et al. (2015).[25] To perform a large number of robustness checks, we use a computationally efficient parallelised slice sampling algorithm. The estimation uses quarterly data for the period 2010q1 to 2019q4.[26] Data for the EU27 are taken from Eurostat. Five variables are used as observables: real output, real consumption, real investment, hours, emissions. All variables have been HP-filtered and expressed as percentage deviations from the HP trend. To avoid stochastic

[24]Being the estimation based on aggregate data the Rep version has been used. We use the Dynare software 4.5 to solve the linearised model and to perform the estimation (see Adjemian et al. (2011)).

[25]The slice sampler algorithm was introduced by Neal (2003). Planas et al. (2015) reconsider the slices along the major axis of the ellipse to better fit the distribution than any Euclidean slices. The slice sampler has been shown to be more efficient and to offer better mixing properties than the Metropolis-Hastings sampler (Calés et al., 2017). Similar Bayesian techniques are used by Giovannini et al. (2019) or Hohberger et al. (2019).

[26]We exclude data for 2020 from the parameter estimation to eliminate noise associated with the exceptional volatility during the COVID crisis.

singularity, we estimate a set of shocks equal to the number of observable variables. The following aggregate shocks are considered: a shock to the Total Factor Productivity (TFP), an emission-intensity shock, a saving shock, an investment shock, a carbon pricing shock.

In Table 2, we report the estimated standard deviation of the shocks considered and their autocorrelation parameters. For the standard deviations, we have elicited an Inverse Gamma distribution with 2 degrees of freedom, while all the AR(1) processes follow a Beta distribution. We report posterior mean, posterior mode and the 90% credible interval.

Table 2: Prior and posterior distribution of aggregate shock parameters

	Prior distribution			Posterior distribution		
Parameter	**Distr.**	**Mean**	**Std.**	**Mode**	**10%**	**90%**
ρ_A	Beta	0.50	0.20	0.9686	0.9211	0.9885
ρ_Ω	Beta	0.50	0.20	0.9644	0.9614	0.9889
ρ_C	Beta	0.50	0.20	0.9754	0.6501	0.9883
ρ_I	Beta	0.50	0.20	0.9654	0.4384	0.9863
ρ_{p_z}	Beta	0.50	0.20	0.3131	0.1126	0.5654
σ_A	Invgamma	0.01	1.00	0.0028	0.0024	0.0035
σ_Ω	Invgamma	0.01	1.00	0.3083	0.2556	0.3649
σ_C	Invgamma	0.01	1.00	0.0083	0.0064	0.0095
σ_I	Invgamma	0.01	1.00	0.0023	0.0016	0.0028
σ_{p_z}	Invgamma	0.01	1.00	0.0024	0.0019	0.0028

Note: Col. (1) lists model parameters. Col. (2) indicates the prior distribution function (Beta: Beta distribution; Invgamma: Inverse Gamma distribution). Cols. (3)-(4) the prior values (mean and standard deviation). Cols. (5)-(7) show the mode and the HPD intervals of the posterior distributions.

The estimated shock persistence (Posterior Mode) is relatively high for almost every shock, which implies a slow propagation of the innovations in the economic system, consistently to the sluggish movement of the observables. The estimated persistence of the policy shock ρ_{p_z} is lower given that less persistent dynamics of the observed emissions. The standard deviation values are close to such similar estimates for Europe. The standard deviation of the emission intensity shock σ_Ω is sizable but nonetheless it plays a minor role in fitting data given the low level of the emission share ζ.

4.3. Micro-Targeted Parameters

The micro-level parameters, shaping the firm-level investment decisions, are derived from a sample of firms subject to the EU Emissions Trading System (EU-ETS), combining information from several sources. The ORBIS balance-sheet data,[27] which harmonizes firm-level data into a global standard format and enables cross-country comparison, is the main data source. We restrict our analysis to NACE Rev. 2 compliant sectors and nations covered by the EU-ETS.[28]

The European Transaction Log (EUTL) registry is the second data source. All regulated plans under the EU-ETS system are listed in this registry. Indeed, we included the installation-based information in EUTL information at the firm level because a company that owns an EU-ETS operator account may have many installations (establishments) subject to the legislation. Additionally, we matched the data in the EUTL on a company level with ORBIS using national identity numbers present in both datasets in order to identify all companies both within ORBIS and subject to the EU-ETS.[29]

Real investment i and real capital k were computed using total fixed asset and depreciation from ORBIS and CPI deflator from the OECD in a Perpetual Inventory Method (PIM), as described in Gal (2013), Andrews et al. (2016) and Gregori et al. (2021). Results are robust when substituting CPI deflator with the investment price deflator, but the latter generated a larger number of missing observations.

The final database, composed of around 45000 firm-year observations, is informative about investment data for firms under the EU-ETS system for the period 2005-2021.

We use the distributional information of regulated firms of this dataset in order to calibrate the micro parameter of the model, given the economic parameters and the estimated aggregate shock. Specifically, we matched two micro parameters, that is the upper bound of the fixed costs $\bar{\psi}$ and the standard deviation of the idiosyncratic productivity shock σ_ϵ, with two targets, that is the standard deviation of the investment rates and the spike rate of investments[30] related to the firms subject to the EU ETS system.[31]

The parameter values are listed in Table 3. They are roughly in the range of previous

[27]ORBIS is maintained by the commercial data provider Bureau van Dijk. The investment data we have extracted by Bureau van Dijk are as of July 2022.

[28]This phase involves fundamental cleaning procedures, such as deleting duplicate observations by ID, year, and accounts consolidation type. This first cleaning procedure follows the guidelines of Bajgar et al. (2020).

[29]In particular, we rely on the linkage between those two datasets established by Letout (2022) directly providing the ORBIS identifier for each operator account in the EUTL. Systematic errors in EUTL have been corrected to make national identifiers compatible with country specific formats as in ORBIS.

[30]Following Cooper and Haltiwanger (2006) for the formal definition of this distributional characteristics, we define *spikes* is the probability that i/k exceeds 0.20. For a further details about the distributional characteristics of regulated firms see Appendix E.

[31]In more details, we generate a quasi-random (quasi-Montecarlo) sequence of initial values to create a grid of pseudo-parameters eligible for the solution of the minimization problem. We then use this grid to run several trials and compute the quadratic deviation of the associated moments from the empirical ones. We then select the parameters associated with the trial that has the smallest residual.

Table 3: Micro-Targeted Parameters

Parameter	Description	Value
$\bar{\psi}$	Upper bound of fixed costs	0.025
σ_ϵ	Standard deviation ϵ	0.036

findings in the literature. Specifically, the upper bound on the fixed cost $\bar{\psi}$ is quite close to the corresponding value as in Khan and Thomas (2008) but somehow lower than Winberry (2021). The calibrated value implies that the average fixed cost paid conditional on adjusting is 3.1 percent of firms' average quarterly output. The standard deviation of the idiosyncratic productivity shock is comparable with previous results in the literature.

4.4. Model Validation

Table 4 compares the matching-targets results from the model with the firm-level empirical targets on annual basis. The Table also depicts some unconditional business statistics for aggregate variables on quarterly basis for both the Het and the Rep model versions.

Table 4: Empirical Targets and Business Cycle Statistics

	Data	Het	Rep
Standard deviation of investment rates	0.407	0.398	-
Spike rate (percent)	0.294	0.283	-
$\sigma(C)/\sigma(Y)$	0.980	0.595	0.490
$\sigma(I)/\sigma(Y)$	3.226	3.512	2.651
$\sigma(N)/\sigma(Y)$	0.360	0.386	0.404
$\sigma(Z)/\sigma(Y)$	2.574	1.931	1.833
$\rho(C,Y)$	0.864	0.896	0.748
$\rho(I,Y)$	0.528	0.931	0.955
$\rho(N,Y)$	0.857	0.610	0.801
$\rho(Z,Y)$	0.306	0.563	0.587

Note: the table reports theoretical moments generated by the model and those of the EU data over the period 2005Q1-2021Q4, retrieved from Eurostat database (https://ec.europa.eu/eurostat/data/database). Series used: Gross domestic product at market prices (chain linked 2010 NAC), Household and NPISH final consumption expenditure (chain linked 2010 NAC), Gross fixed capital formation (chain linked 2010 NAC), Hours worked, Emissions (Greenhouse gas emission statistics - air emissions accounts).

The model is able to capture the frequency of spikes which is informative about the strength of fixed costs. The model also captures the dispersion of investment rates across firms, which is informative about the size of idiosyncratic shocks. In matching the micro-targets, it nonetheless replicates the business cycle statistics fairly well despite the fact that the model does not have price stickiness, habit formation and adjustment costs. Investment is more volatile than output and consumption is less volatile than output in both the model and the data. The volatility of hours is very close to the data, while the volatility of emissions are higher than output. This feature is consistent with some previous empirical studies about business cycle and emissions (see Doda (2014)). Overall, the predicted fluctuations in macroeconomic aggregates appear reasonably close to the data, including the fit of emission dynamics.[32]

[32] Within conventional E-DSGEs fitting the emission dynamics model remains relatively more challenging.

5. Business Cycle and Environmental Policy

To elucidate the role of heterogeneity in shaping the relationship between business cycle and environmental policy, in this section we analyze the transmission of several aggregate shocks under two alternative model versions: the model with heterogeneous firms described above (Het model), the corresponding representative agent model (Rep model).

The comparison of the two model versions allows us to focus on the distributional effects stemming from the the business cycle shocks conditional to the environmental regime. Also, we analyse how different shocks might draw different policy implications depending on the source of the business cycle and on the model in use.

We examine the effects of five shocks typically associated to the business cycle and the environmental policy: (i) a positive shock to the Total Factor Productivity (TFP); (ii) a temporary emission-intensity shock that is assumed to capture technological change in abatement; (iii) a temporary shock on consumption; (iv) a temporary shock on investment; (v) a temporary permit price increase: in this case we assume that the carbon price is an exogenous process of the type[33]:

$$p_{z,t} = \rho_{z,t} p_{z,t-1} + \epsilon_{z,t} \tag{32}$$

We also assume that the policy remains unchanged independently of the effects of the regulation on the initial size of firm distribution. The auto-correlation and standard deviation of the exogenous shocks are those estimated as reported in Table 2. To have a clear comparison of the dynamics in the two model-version responses, we mostly highlight a decomposition of the demand-side components (consumption, investment and abatement costs) into the effects of variables that condition the firm-level dynamics. Such variables include interest rate, wages and permit price.

We show the key variables dynamics (namely, consumption, investment and abatement costs) in the two models as share of their respective output and reported as percentage points from the initial steady state per unit of their respective output over a 20-quarter period. This allows us to disentangle the overall demand adjustment into its component variations.

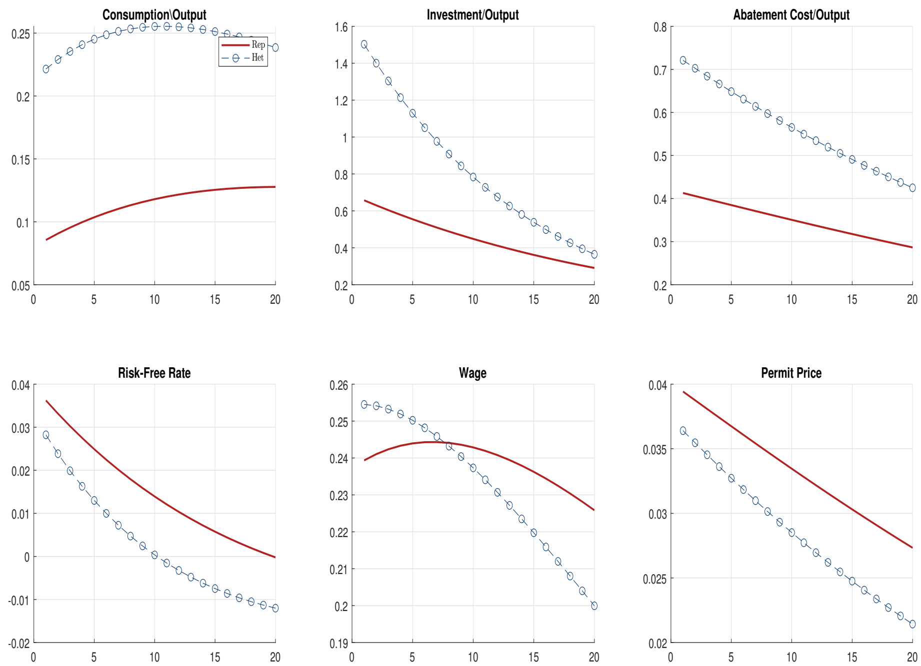
All the simulation results for the competitive economy have been obtained by using a 'pure' perturbation method which amounts to a first-order Taylor approximation of the model around its deterministic steady state.

[33] It is worth noticing that for this shock we assume the environmental regime at play is different from the Cap and Trade since emissions are assumed variable while the carbon price follows an exogenous process.

5.1. TFP shock

The impulse response to a 0.28% percent increase in productivity A_t is shown in Figure 1 for both the Het (dashed, blue lines) and the Rep model (continuous, red lines). Firms are subject to the cap-and-trade regime.

Figure 1: Impulse responses to the TFP shock



Note: Horizontal axis: quarters. Vertical axis: Consumption, Investment and Abatement Cost are expressed as pp variation from the steady state over output ($\frac{\Delta X}{Y}$ *100). Estimated persistence of the shock is 0.97 and its std is 0.28%. Wage and permit price are expressed as percentage deviations from the initial steady state, while the interest rate is expressed in percentage-point deviations).

In response to a positive TFP shock, the two model versions show a similar qualitative response. Output increases persistently. Because household wealth grows as a result of increased production, consumption follows suit. As a matter of fact, consumption exhibits an inverted U-shaped pattern in bot the two model versions, with positive reactivity both on impact and in subsequent periods. Furthermore, because of the lower marginal costs, expanding production firms find it advantageous to invest in new capital for future production. The increase in productivity calls for an higher demand of emissions and thus a strong reaction of the abatement effort in order to comply with the emission regulation. This is why both abatement costs and emission price go up.

Nonetheless, the response of the two model versions is quantitatively different. In particular, in the Het model we see that these effects are amplified for all the demand components especially for investment.

This is due to the different transmission of the shock in the two models. In particular, two channels account for the major differences across the model versions.

First, since the elasticity of investment with respect to the shock is procyclical, in the Het model an expansionary impulse will push more firms to make an extensive margin investment, basically changing the distribution of it. This means that, as a result of this shock, a larger mass of firms will increase production capacity so that the number of firms making new investments increases on impact. This adds to the aggregate shock, making it more reactive. The fixed costs, associated to the wage, slow down the way in which firms make margin decision about whether to invest or not (see equation (26)). This is why investment is shown to decrease more rapidly then the Rep version, as the shock fades away.

Second, the observed adjustment in the real interest rate is different. Interest rate dynamics is highly correlated with productivity in the Rep model, implying a stronger procyclical movement. Indeed, in the Rep model the extreme sensitivity of the demand side to changes in interest rates is capable of bringing them in line with household preferences and output variation. In this sense the interest rate plays the major role in ensuring then zero profit condition to the representative firm.[34] On the contrary, in the Het model, the interest rate's response to the productivity shock is weaker than in the Rep case. Higher consumption response is thus caused by the fact that, in the Het model, interest rate does not dampen the cyclical movement of consumption demand, thereby magnifying its reaction. Furthermore, the weaker reaction of the interest rate makes the abatement activity more convenient than buying emission permits.

In this case, the firm heterogeneity does not translate into an aggregate efficiency gain for firms that are able to comply with the environmental target (the emission Cap) with more abatement effort and thus with higher abatement costs.

The positive response of wage in response to productivity shock is due to the intertemporal substitution effects of the interest rate as in the typical RBC model.[35] In this respect we notice that the response of wages in the Het model is stronger in impact and weaker in the subsequent periods.

All in all, as a result of the productivity shock, the presence of heterogeneous firms and

[34] The zero profit condition is a consequence of constant returns of scale in the production function. This condition is typically assumed in a perfect competition setup, see for instance Rebelo (2005).

[35] See King and Rebelo (1999) for more details.

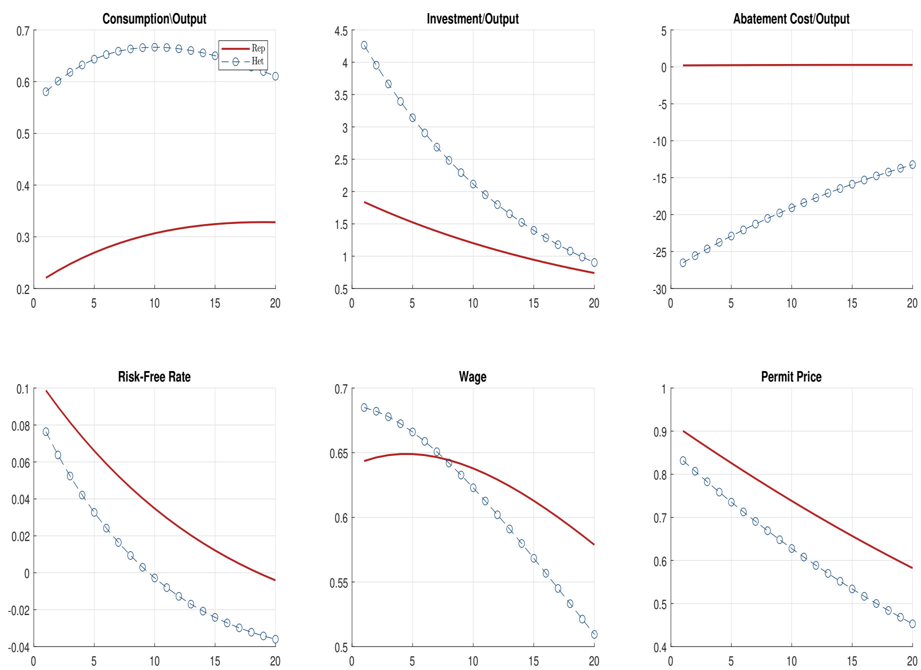
environmental policy makes consumption, investment and abatement decisions more volatile than in the Rep model.

5.2. *Emission-intensity shock*

Figure 2 displays the impulse response to a positive change in Ω_t (see the shock description in (11)).

Overall, the macroeconomic effects of this shock resembles to the TFP, since we are assuming a technological shift but specific to the abatement function. More precisely, this is a cost-reducing technical change in abatement, implying that higher Ω_t decreases the abatement cost across all firms.

Figure 2: Impulse responses to the Emission-Intensity shock



Note: Horizontal axis: quarters. Vertical axis: Consumption, Investment and Abatement Cost are expressed as pp variation from the steady state over output ($\frac{\Delta X}{Y}$ *100). Estimated persistence of the shock is 0.96 and its std is 31%. Wage and permit price are expressed as percentage deviations from the initial steady state, while the interest rate is expressed in percentage-point deviations).

According to the theory, the current shock will lead to a technical advancement in emissions reduction. This advancement enables firms to achieve a more efficient emissions reduction process, resulting in reduced emissions per unit of output. In other words, the transmission of the shock is induced by a change in the relationship between output and emissions as dictated in equation (10).

As a consequence of this technological breakthrough, there are two opposing effects at work.

The first concern the fact that firms experience an immediate and lasting impact on their output levels due to lower marginal cost of firms. To maintain this higher level of economic activity, they must abate more, thus incurring more costs.

The second, acting on the opposite direction, is related to the fact that now firms are more efficient, able to create more goods and services while also meeting the emission target. As a result, they require a lesser cost of abatement.

While the first effect prevails in aggregate in the Rep model (positive, although mild, abatement cost response), the second effect is predominant in the Het model. This indicates that the shock results in an efficiency gain for heterogeneous firms, because every firm can now comply with the emission Cap with less abatement effort and more installed capital, that is, there will be a greater number of firms with more output, fewer emissions (per capita) and more capital.[36] Given the emission price variation, the improved relationship between emission and output implies less need of abating (see equation (20)). As a result, they require less pollution permits than the equivalent Rep model. The price of emissions rises accordingly.

Given the various adjustments in abating, the corresponding movement of consumption and investment over output shows some significant differences. As for the TFP shock, in the Het model the responsiveness of investment is stronger than in the Rep version, being procyclical to the shock.

Nonetheless, the presence of the fix cost on investment will make the decline of investment sharper as the effect of the shock fades away. The more limited interest rate excursion, in turn, allows for a significant expansion of consumption.

In the Rep mode, the upwards responsiveness of interest rate enables on the one hand the adjustment of all the variables compatible, on the other hand it slows down their reaction.

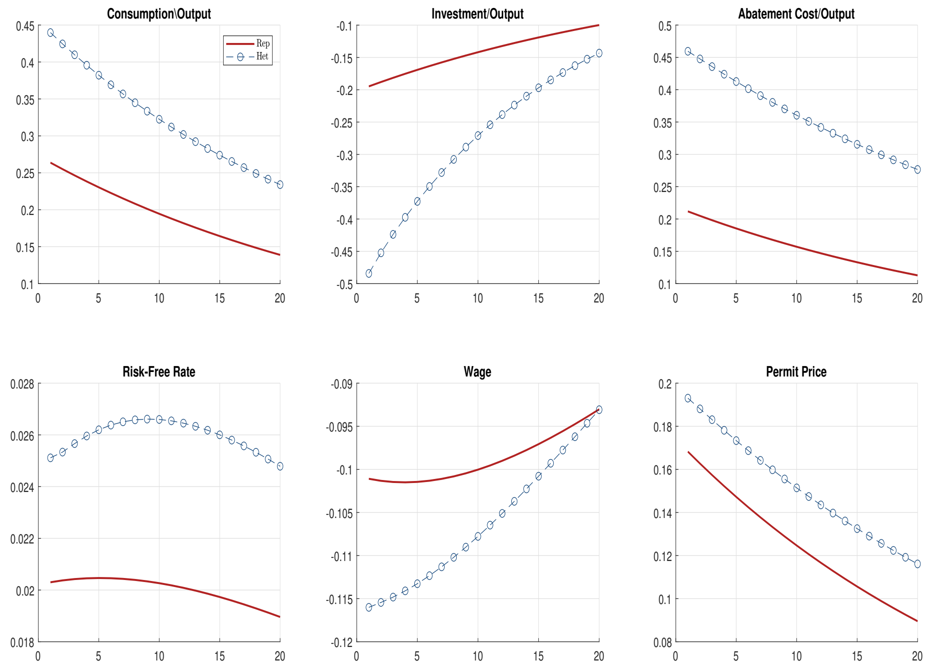
5.3. Consumption Shock

Figure 3 displays the impulse response to an increase in $\Lambda_{c,t}$, respectively, under the Het and the Rep model.

[36]Given that each of them has a specific productivity (due to the idiosyncratic productivity shock $\epsilon_{i,t}$) and capital, this shock implies a substantial shift in the distribution of emissions across heterogeneous firms.

The shock propagation mechanism works similarly to the classic demand shock, impacting consumption and consequently saving exogenously (see equation (16)). These effects are qualitatively similar in the two model versions.

Figure 3: Impulse responses to the Consumption shock



Note: Horizontal axis: quarters. Vertical axis: Consumption, Investment and Abatement Cost are expressed as pp variation from the steady state over output ($\frac{\Delta X}{Y}$ *100). Estimated persistence of the shock is 0.98 and its std is 0.83%. Wage and permit price are expressed as percentage deviations from the initial steady state, while the interest rate is expressed in percentage-point deviations).

More precisely, the shock causes a positive consumption reaction, resulting in a negative saving behavior. Higher consumption raises production, causing households to supply more labor to meet the higher necessary demand. By definition, the shock causes a decline in investments, which is a crowded-out effect caused by the higher propensity to consume. Given that firms must comply with environmental regulations while expanding output, the abatement effort and permit price react quickly, leading abatement costs to further reduce investment. To equalize the inter-temporal marginal rate of substitution with the present consumption price, the real interest rate is rising. This, in turn, tends to slow down consumption expansion. Nonetheless, some substantial differences are observable in the two model versions, particularly in the demand-side adjustment and the accompanying interest rate dynamics. Three specific issues stand out to illustrate the impact of heterogeneity in affecting the adjustment of aggregate variables.

First, the substantial decrease in investments in the Het model is attributable to a large number of firms lowering their capital endowments. In this view, the shock denotes a downward shift in firm capital distribution. According to this viewpoint, firm-level heterogeneity amplifies the negative investment reaction to the shock, which implies that a higher number of firms will not undertake new investment projects.

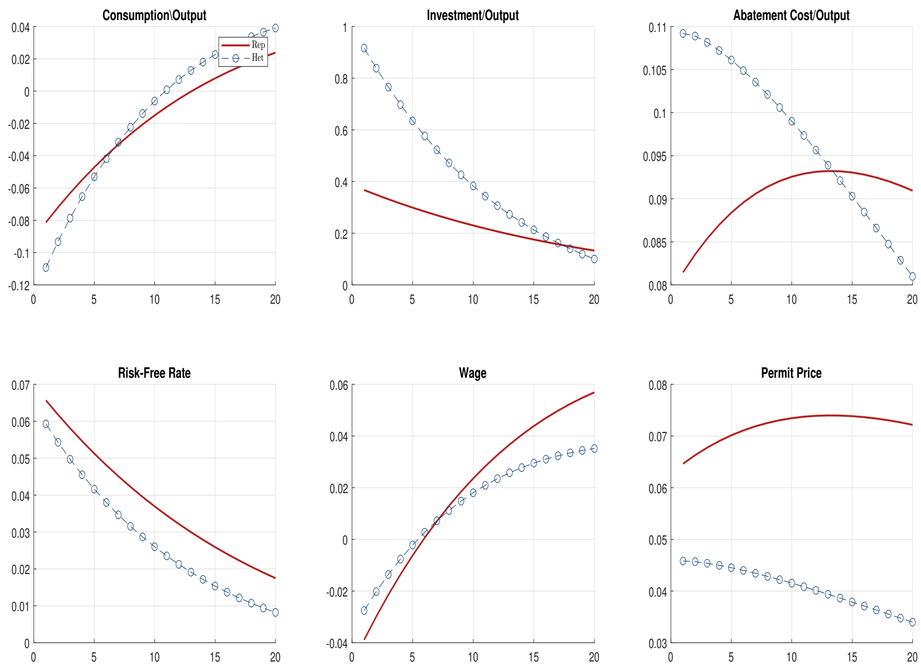
Second, interest rate variation in the Rep model is far lower than in the analogous Het model. This is because the demand components (consumption, investment, and abatement costs) are extremely sensitive to the interest rate movements. This means that a little change in it is able to bring the demand variables in line with the demand components. In this sense, a stronger adjustment in interest rates is required in the Het model to make household and firm decisions consistent with the muted economic conditions.

Third, because a larger number of firms are abating more, the aggregate abatement effort reaction is clearly stronger than the Rep equivalent. The shock implicitly causes an intertemporal replacement of investment with less expensive abatement operations. Firms increase production to accommodate rising abatement expenses. Finally, the permit price increases to comply with the emission Cap accordingly.

5.4. Investment shock

We now focus on the economy's response to a positive shock on investment $\Lambda_{q,t}$. See Figure 4.

Figure 4: Impulse responses to the Investment shock



Note: Horizontal axis: quarters. Vertical axis: Consumption, Investment and Abatement Cost are expressed as pp variation from the steady state over output ($\frac{\Delta X}{Y}$ *100). Estimated persistence of the shock is 0.96 and its std is 0.23%. Wage and permit price are expressed as percentage deviations from the initial steady state, while the interest rate is expressed in percentage-point deviations).

This shock represents an exogenous disturbance to the process by which investment goods are transformed into installed capital to be used in production. See Justiniano et al. (2011) for more details. In other terms, this shock captures the extent to which resources allocated to investment increase the capital stock available to firms for use in production in the next period. As a result, it causes a shock to the marginal efficiency of investment, affecting the productivity of new installations while leaving the productivity of existing capital stock untouched. The shock, in general, captures disruptions to the capital accumulation process. For example, the disruptions might be connected to funding, the legal environment, or even changes in weather conditions that could impair the installation procedure. It may also represent the reality that new technologies are frequently embraced through investment.

The shock raises the rate of return on investment by changing the marginal efficiency of capital. In fact, both model versions show a rise in interest rates. As a consequence, households shift demand away from consumption towards investment as part of a process of intertemporal substitution. The effect on consumption is quantitatively similar in the two model versions.

The reduction in consumption is caused by a shift in the labor supply curve, which results in a wealth impact on the supply side. Because capital is predetermined, the economy shifts down the labor demand curve as a result of the shock. As a result, while consumption falls, hours are increased in order to generate more investments (see Barro and King (1984) for a complete exposition of co-movement of consumption and hours for this shock).

Nonetheless, more substantial differences are observed in the investment response.

Aggregate investment in the Het model responds substantially to the shock as a consequence of two contemporaneous effects: improved marginal efficiency of capital (which is in common with the Rep) and a change in the mass of firms undertaking new investments, that is specific and in place only for the Het version. This is why we observe that the volatility of investment is higher in the Het than in the Rep.

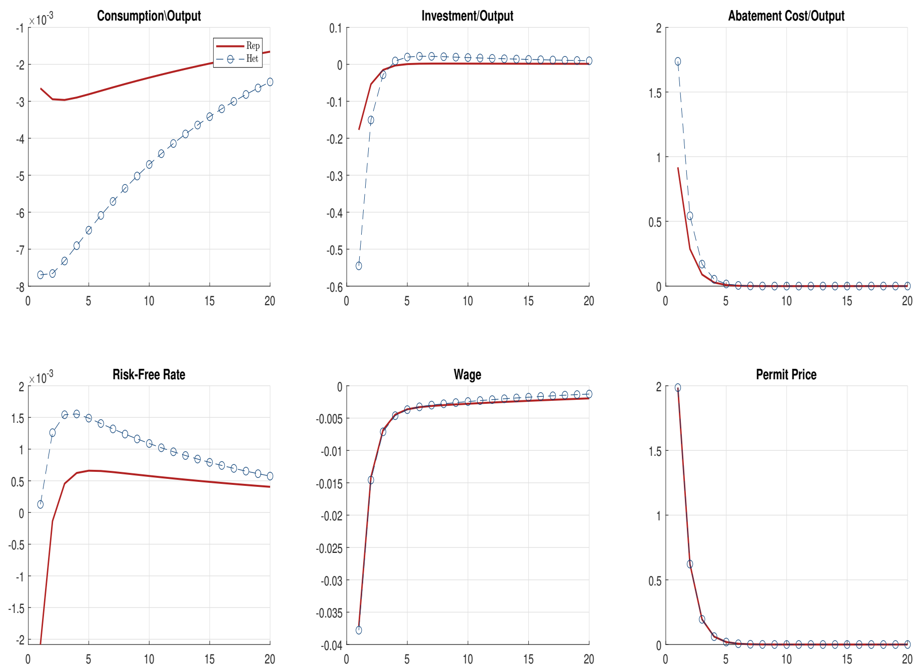
Furthermore, because abatement is now less expensive, heterogeneous firms will find it more advantageous to abate more than in the Rep, therefore carrying increased abatement costs. This enables them comply with emission Cap while still accumulating further capital for future production.

5.5. *Emission Price shock*

We now turn to the impulse response to an increase in the emission price $p_{z,t}$ (see equation (32)). This carbon policy shock could be interpreted as events concerning the supply of emission allowances or similar regulatory events.[37]

Figure 5 shows the macro-responses from this shock for the Het and Rep models.

Figure 5: Impulse responses to the Emission Price shock



Note: Horizontal axis: quarters. Vertical axis: Consumption, Investment and Abatement Cost are expressed as pp variation from the steady state over output ($\frac{\Delta X}{Y}$ *100). Estimated persistence of the shock is 0.31 and its std is 0.24%. Wage and permit price are expressed as percentage deviations from the initial steady state, while the interest rate is expressed in percentage-point deviations).

Equation (20) describes the main relevant mechanism concerned with this shock, where the cost savings associated with reduced emission permit expenditures, $p_{z,t}z_{i,t}$, equate to the equivalent marginal cost, $\zeta y_{i,t}$. The rise in $p_{z,t}$ thus makes emissions more expensive driving firms to cut them by undertaking abating activities. As a result, the abatement effort in both the model configurations is growing.

[37] Alternatively, it might be portrayed, as a shift in the emission Cap given the inverse relationship between permit price and aggregate emissions. It is worth noticing that for this shock emissions are assumed variable while the carbon price follows the process described in (32).

Nonetheless, this increased price of emissions entails a shift in marginal costs of firms, so that all the demand components (consumption, investment and abatement costs) adjust down according to the drop in production on both the model versions. In particular, according to the this negative output variation, firms decide to restrict capital accumulation. However, some quantitative differences stand out for all the variables in the two model versions.

The nonlinearity of investment decisions drives the crowding-out effect in the Het model. Given the negative investment decisions caused by increased marginal costs, on impact a higher number of enterprises will limit capital accumulation, amplifying investment variation. Given that the Het model implies that small firms (firms with low capital) are less efficient in abating, a higher abatement cost is now needed to comply with the higher $p_{z,t}$. So that, overall, in the Het version we observe a sharp decline of investment and a rapid increase of abatement costs on impact due to distributional issues, namely firms with lower capital endowment and lesser efficient in abating.

Turning to consumption, we observe a stronger drop in the Het than in the Rep version. Given the quick variation in investment dynamics, in the Het model the concurrent increase in interest rates magnifies the decrease in consumption, making the demand and supply decisions compatible. Indeed, we observe that, on impact, the interest rates follow an opposite path in the two models, jumping up for the Het and adjusting down for the Rep.

6. Conclusions

We present a RBC model embodying incorporating environmental regulation and heterogeneous firms. We analyse the interplay between environmental policy and economic uncertainty when when firm-level productivity is heterogeneous, and firms make extensive-margin investment decisions. The paper highlights the relevance of firm heterogeneity in accounting for the macroeconomic volatility showing that the business cycle fluctuations are accentuated in presence of firms undertaking extensive-margin investment decisions, resulting in a different dynamic adjustment relative to the Representative agent counterpart. This conclusion holds regardless of the origin of the shocks.

In particular, we examine the dynamic response of the economy to five shocks under a Cap-and-Trade regime, namely two supply-side shocks (TFP and an emission-intensity shock), two demand shocks (saving and investment shocks), and a emission price shock. We show that the shocks tend to widen demand side volatility in comparison to the equivalent representative-agent model, in that heterogeneity plays a crucial role in shaping investment dynamics when the shocks originate from the demand side. The Emission Price shock highlights the aggregate implications of investments with heterogeneous firms. We argue that the insights gained here pave the way for more extensive investigations into the link between the business cycle and environmental policy with heterogeneous agents.

Incorporating more frictions into the study might be an important path for future research, because ideally these types of frictions are likely to interplay with firm heterogeneity and thus to magnify the sources of uncertainty that are major drivers of economic volatility. Frictions of this type are, for examples, the price stickiness, labor market frictions in the form of wage rigidities and labor adjustment costs.

Finally, the methodology in this study only allows us to investigate the effects of environmental policy and offer policy recommendations for an economy that has attained its steady-state level. However, because climate change environmental policy focuses on long-term objectives, more work should be devoted to the study of the economy's transition toward a low-carbon economy, beginning accounting for uncertainty and heterogeneity, and comparing the performance of the different policy options along the adjustment path.

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Appendix A.

The current aggregate state is $s = \left(A, \Omega, \Lambda_q, \Lambda_c, g\right)$ The firm-level equilibrium conditions describing the economy are the following:

$$y(\epsilon, k; s) = \left(\phi e^{\Omega} z\right)^{\zeta} \left(\Delta(s) e^{A} e^{\epsilon}\right)^{1-\zeta} n(\epsilon, k; s)^{\nu} k^{\theta}. \tag{A-1}$$

$$w(s) = \nu \left(\phi e^{\Omega} z\right)^{\zeta} \left(\Delta(s) e^{A} e^{\epsilon}\right)^{1-\zeta} n(\epsilon, k; s)^{\nu-1} k^{\theta} \tag{A-2}$$

$$p_Z(s) = \zeta \left(z\right)^{\zeta-1} \left(\phi e^{\Omega}\right)^{\zeta} \left(\Delta(s) e^{A} e^{\epsilon}\right)^{1-\zeta} n(\epsilon, k; s)^{\nu} k^{\theta} \tag{A-3}$$

$$z(\epsilon, k; s) = (1 - u(\epsilon, k; s)) \frac{q(\epsilon, k; s)}{\phi e^{\Omega}}, \tag{A-4}$$

$$q(\epsilon, k; s) = \Delta(s) e^{A} e^{\epsilon} n(\epsilon, k; s)^{\alpha} k^{\beta}, \tag{A-5}$$

$$\frac{\lambda(s)}{e^{\Lambda_q'}} = \beta \lambda(s') \left(\phi e^{\Omega' z(\epsilon', k'; s')}\right)^{\zeta} \left(\Delta(s') e^{A'} e^{\epsilon'}\right)^{1-\zeta} n(\epsilon', k'; s')'^{\nu} \theta k'^{\theta-1} + \frac{(1-\delta)}{e^{\Lambda_q'}} \tag{A-6}$$

The aggregate equilibrium conditions describing the economy are the following:

$$\lambda(s) = e^{\Lambda_c'} C(s)^{-\gamma} \tag{A-7}$$

$$w(s) = \frac{\chi N^{\varphi}(s)}{\lambda(s)} \tag{A-8}$$

$$1 + r(s) = \frac{\lambda(s)}{\beta \lambda(s')} \tag{A-9}$$

$$M(s') = Z(s) + Z(s)^{RoW} + (1 - \delta_M) M(s), \tag{A-10}$$

$$\Delta(s) = e^{-\chi(M(s) - \tilde{M})}, \tag{A-11}$$

$$T(s) = p_Z Z(s) \tag{A-12}$$

$$I\left(s\right) = \int p_{\psi}(\epsilon, k; s) \left[k_{\{i \neq 0\}}(\epsilon, k; s) - \left(1 - \delta_k\right) k\right] g\left(\epsilon, k\right) d\epsilon dk \tag{A-13}$$

$$N\left(s\right) = \int \left[n\left(\epsilon, k; s\right) + n_{\psi}(\epsilon, k; s)\right] g\left(\epsilon, k\right) d\epsilon dk \tag{A-14}$$

$$Z\left(s\right) = \int z\left(\epsilon, k; s\right) g\left(\epsilon, k\right) d\epsilon dk \tag{A-15}$$

$$Y(\boldsymbol{s}) = \int \left(\phi e^{\Omega} z(\epsilon, k; \boldsymbol{s})\right)^{\zeta} \left(e^{A} e^{\epsilon} \Delta(\boldsymbol{s})\right)^{1-\zeta} n(\epsilon, k; \boldsymbol{s})^{\nu} k^{\theta} g(\epsilon, k)\, d\epsilon dk \tag{A-16}$$

$$Y(\boldsymbol{s}) = C(\boldsymbol{s}) + I(\boldsymbol{s}) \tag{A-17}$$

$$AC(\boldsymbol{s}) = \int \left[1 - (1 - u(\epsilon, k; \boldsymbol{s}))^{\zeta}\right] q(\epsilon, k; \boldsymbol{s}) d\epsilon dk \tag{A-18}$$

$$Q(\boldsymbol{s}) = Y(\boldsymbol{s}) + AC(\boldsymbol{s}) \tag{A-19}$$

$$\epsilon' = \rho_{\epsilon} \epsilon + \epsilon'_{\epsilon} \tag{A-20}$$

$$A' = \rho_{A} A + \epsilon'_{A} \tag{A-21}$$

$$\Omega' = \rho_{\Omega} \Omega + \epsilon'_{\Omega} \tag{A-22}$$

$$\Lambda'_{q} = \rho_{\Lambda_q} \Lambda_q + \epsilon'_{\Lambda_q} \tag{A-23}$$

$$\Lambda'_{c} = \rho_{\Lambda_c} \Lambda_c + \epsilon'_{\Lambda_c} \tag{A-24}$$

Appendix B.

The equilibrium conditions describing the Representative agent model are the following:

$$Y_t = \left(\phi e^{\Omega_t} Z_t\right)^{\zeta} \left(\Delta_t e^{A_t}\right)^{1-\zeta} N_t^{\nu} K_t^{\theta}. \tag{A-25}$$

$$w_t = \nu \left(\phi e^{\Omega_t} Z_t\right)^{\zeta} \left(\Delta_t e^{A_t}\right)^{1-\zeta} N_t^{\nu-1} K_t^{\theta} \tag{A-26}$$

$$p_{Z,t} = \zeta \left(Z_t\right)^{\zeta-1} \left(\phi e^{\Omega_t}\right)^{\zeta} \left(\Delta_t e^{A_t}\right)^{1-\zeta} N_t^{\nu} K_t^{\theta} \tag{A-27}$$

$$\frac{\lambda_t}{e^{\Lambda_{q,t}}} = \beta \lambda_{t+1} \left(\phi e^{\Omega_{t+1}} Z_{t+1}\right)^{\zeta} \left(\Delta_{t+1} e^{A_{t+1}}\right)^{1-\zeta} N_{t+1}^{\nu} \theta K_{t+1}^{\theta-1} + \frac{(1-\delta)}{e^{\Lambda_{q,t+1}}} \tag{A-28}$$

$$Z_t = (1-u_t) \frac{Q_t}{\phi e^{\Omega_t}} \tag{A-29}$$

$$Q_t = \Delta_t e^{A_t} N_t^{\alpha} K_t^{1-\alpha} \tag{A-30}$$

$$\lambda_t = e^{\Lambda_{c,t}} C_t^{-\gamma} \tag{A-31}$$

$$w_t = \frac{\chi N^{\varphi}}{\lambda_t} \tag{A-32}$$

$$1 + r_t = \frac{\lambda_t}{\beta \lambda_{t+1}} \tag{A-33}$$

$$M_{t+1} = Z_{t+1} + Z_{t+1}^{RoW} + (1-\delta_M) M_t, \tag{A-34}$$

$$\Delta_t = e^{-\chi(M_t - \tilde{M})}, \tag{A-35}$$

$$T_t = p_{Z,t} Z_t \tag{A-36}$$

$$Y_t = C_t + I_t \tag{A-37}$$

$$AC_t = \left[1 - (1-u_t)^{\zeta}\right] Q_t \tag{A-38}$$

$$Q_t = Y_t + AC_t \tag{A-39}$$

$$A_{t+1} = \rho_A A_t + \epsilon_{A,t+1} \tag{A-40}$$

$$\Omega_{t+1} = \rho_{\Omega}\Omega_t + \epsilon_{\Omega,t+1} \tag{A-41}$$

$$\Lambda_{q,t+1} = \rho_{\Lambda_q}\Lambda_{q,t} + \epsilon'_{\Lambda_q,t+1} \tag{A-42}$$

$$\Lambda_{c,t+1} = \rho_{\Lambda_c}\Lambda_{c,t} + \epsilon'_{\Lambda_c,t+1} \tag{A-43}$$

Appendix C.

In the optimization problem, we define

$$\tilde{\psi}(\epsilon,k;\mathbf{s}) = \frac{v_{\{i\neq 0\}}(\epsilon,k;\mathbf{s}) - v_{\{i=0\}}(\epsilon,k;\mathbf{s})}{\lambda(\mathbf{s})w(\mathbf{s})}, \quad \hat{\psi}(\epsilon,k;\mathbf{s}) = \min\{\tilde{\psi}(\epsilon,k;\mathbf{s}),\bar{\psi}\}, \quad p_{\psi}(\epsilon,k;\mathbf{s}) = \frac{\hat{\psi}(\epsilon,k;\mathbf{s})}{\bar{\psi}}.$$

Condition (26) is verified if and only if $\psi \leq \hat{\psi}$, then:

$$\begin{aligned}
\hat{v}(\epsilon,k;\mathbf{s}) &= \left\{\left(\phi e^{\Omega} z(\epsilon,k;\mathbf{s})\right)^{\zeta} \left(e^{A} e^{\epsilon}\Delta(\mathbf{s})\right)^{1-\zeta} n(\epsilon,k;\mathbf{s})^{\nu} k^{\theta} - w(\mathbf{s})n(\epsilon,k;\mathbf{s}) - p_z z(\epsilon,k;\mathbf{s})\right\} \\
&+ p_{\psi}(\epsilon,k;\mathbf{s})\left(v_{\{i\neq 0\}}(\epsilon,k;\mathbf{s}) - \frac{\hat{\psi}}{2}\lambda(\mathbf{s})w(\mathbf{s})\right) + \left(1 - p_{\psi}(\epsilon,k;\mathbf{s})\right) v_{\{i=0\}}(\epsilon,k;\mathbf{s}) \\
&= \left\{\left(\phi e^{\Omega} z(\epsilon,k;\mathbf{s})\right)^{\zeta} \left(e^{A} e^{\epsilon}\Delta(\mathbf{s})\right)^{1-\zeta} n(\epsilon,k;\mathbf{s})^{\nu} k^{\theta} - w(\mathbf{s})n(\epsilon,k;\mathbf{s}) - p_z z(\epsilon,k;\mathbf{s})\right\} \\
&+ p_{\psi}(\epsilon,k;\mathbf{s})\left(-\lambda(\mathbf{s})\left(k_{\{i\neq 0\}}(\epsilon,k;\mathbf{s}) - (1-\delta_k)\,k\right)\right. \\
&+ \left.\beta E\left[\hat{v}(\epsilon',k_{\{i\neq 0\}}(\epsilon,k;\mathbf{s}),\mathbf{s'})|\epsilon,k;\mathbf{s}\right] - \lambda(\mathbf{s})w(\mathbf{s})\hat{\psi}\left(\epsilon,k;\mathbf{s}\right)\right)/2\Big) \\
&+ \left(1 - p_{\psi}(\epsilon,k;\mathbf{s})\right)\beta E\left[\hat{v}(\epsilon',(1-\delta_k)\,k,\mathbf{s'})|\epsilon,k;\mathbf{s}\right]
\end{aligned} \tag{A-44}$$

Furthermore, $p_{\psi}(\epsilon,k;\mathbf{s})$ can be interpreted as the probability of investing or as the fraction of the firms that invest. The expected cost in number of hours that should be spent for investing of firms with productivity shock ϵ and capital k is:

$$n_{\psi}(\epsilon,k;\mathbf{s}) = \int_0^{\hat{\psi}(\epsilon,k;s)} \psi \frac{d\psi}{\bar{\psi}} = \frac{\hat{\psi}\left(\epsilon,k;\mathbf{s}\right)^2}{2\bar{\psi}}. \tag{A-45}$$

Appendix D.

Most DSGE models assume that all changes in aggregate investment are the result of a representative firm operating along the intensive margin. This is consistent with constant returns of scale implying that investment is nearly infinitely sensitive to changes in the interest rate.

In the presence of heterogeneous firms and lumpy investment, constant return to scale result in an infinite elasticity of investment at the intensive margin with respect to the risk free interest rate.

As an example, if the production function is of the type:

$$q_{i,t} = e^{A_t} e^{\epsilon_{i,t}} k_{i,t}^{\beta} \tag{A-46}$$

the semi-elasticity with respect to the real interest rate is:

$$\frac{1}{i_{i,t}} \frac{\partial i_{i,t}}{\partial r_t} = -\frac{1}{\delta} \frac{1}{1-\beta} \left(\frac{1+r_t}{r_t+\delta} \right) \tag{A-47}$$

in which the constant return of scale ($\beta = 1$) would result in an infinite elasticity of investment relative to the interest rate.

For more details see Winberry (2021) and the references therein.

Appendix E.

Table 5: Target Distributional Characteristics (standard errors in parenthesis)

	i/k
Mean	0.217 (0.002)
Std	0.407 (0.009)
Inaction	0.057 (0.001)
Spikes	0.294 (0.002)
Negative Spikes	0.014 (0.001)

Table 5 summarizes those distributional characteristics including the mean and the standard deviation of the investment rate i/k. We will use some of these distributional information as a target for calibrating the relevant micro parameters in the model.

Figure 6: Distribution of the investment rate $\frac{i}{k}$

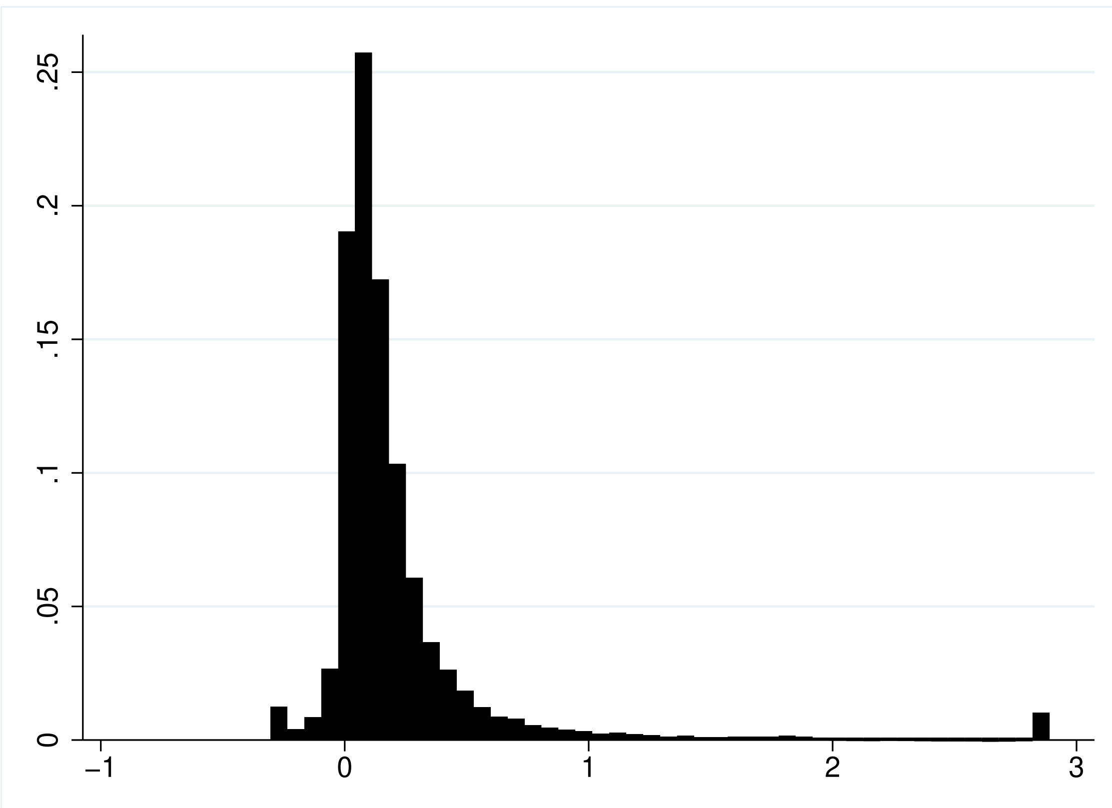

Figure 1 shows the distribution of i/k featuring the the usual characteristics of irreversible and lumpy investment as in Cooper and Haltiwanger (2006). In short, the fat right tail comes from substantial positive investments in a single year (*spikes*) while the asymmetry between the left and the right tail is a symptom of partial irreversibility, that is high cost in large disinvestments (*negative spikes*). Furthermore, the large mass concentrated around zero indicates firms that are not investing (*inaction*). Following Cooper and Haltiwanger (2006) for the formal definition of this distributional characteristics, we define *inaction* as the probability that i/k in absolute value is less or equal than 0.01, *spikes* is the probability that i/k exceeds 0.20 and *negative spikes* is the probability that i/k is less than -0.20.