DOMANDA: Quanto vale il prezzo di ZCB in questo modello?

$$p(t,T) = F^T(t, r_t) \quad \text{t.c.}$$

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + a(\gamma - r_t)\dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2 r_t \dfrac{\partial^2 F^T}{\partial r^2} - r F^T = 0 \\ F(T, r_T) = 1 \end{cases}$$

(PROP. IN HO-LEE) $\Rightarrow$

$$p(t,T) = \exp\{A(t,T) - B(t,T) r\},$$

con $\alpha_t = -a$, $\beta_t = a\gamma$, $\gamma_t = \sigma^2$, $\delta_t = 0$

$$\Rightarrow \begin{cases} \dot{A}(t,T) + a\gamma B(t,T) = 0, & A(T,T) = 0 \\ \dot{B}(t,T) - a B(t,T) - \frac{1}{2}\sigma^2 (B(t,T))^2 + 1 = 0, & B(T,T) = 0 \end{cases}$$

$\hookrightarrow$ ODE DI RICCATI (Eq. diff. ordinaria, del primo ordine, quadratica)

$$\tau := T - t \Rightarrow \dot{B}(\tau) = \frac{1}{2}\sigma^2 B^2(\tau) + a B(\tau) - 1, \quad B(0) = 0$$

$$\Rightarrow \sigma^2 B^2 - \dot{B} = -aB + 1$$

GUESS AND VERIFY APPROACH: $B(\tau) = -\dfrac{2}{\sigma^2}\dfrac{y'(\tau)}{y(\tau)}$, $y(\tau) \neq 0$ (*)

$$\Rightarrow \dot{B}(\tau) = -\frac{2}{\sigma^2}\frac{y''\cdot y - (y')^2}{y^2} = -\frac{2}{\sigma^2}\frac{y''}{y} + \frac{2}{\sigma^2}\frac{(y')^2}{y^2}$$

$$\overset{(*)}{\Rightarrow} -\frac{2}{\sigma^2}\frac{y''}{y} + \frac{2}{\sigma^2}\frac{(y')^2}{y^2} = \frac{\sigma^2}{2}\left(-\frac{2}{\sigma^2}\frac{y'}{y}\right)^2 + a\left(-\frac{2}{\sigma^2}\frac{y'}{y}\right) - 1$$

$$\Rightarrow -\frac{2}{\sigma^2}\frac{y''}{y} + \frac{2}{\sigma^2}\frac{(y')^2}{y^2} = \frac{\cancel{\sigma^2}}{\cancel{2}}\frac{\overset{2}{\cancel{4}}}{\sigma^{\cancel{4}\,2}}\frac{(y')^2}{y^2} - \frac{2a}{\sigma^2}\frac{y'}{y} - 1$$

$$\Rightarrow -\frac{2}{\sigma^2}\frac{y''}{y} + \cancel{\frac{2}{\sigma^2}\frac{(y')^2}{y^2}} = \cancel{\frac{2}{\sigma^2}\frac{(y')^2}{y^2}} - \frac{2a}{\sigma^2}\frac{y'}{y} - 1$$

$$\Rightarrow -\frac{2}{\sigma^2}\frac{y''}{y} + \frac{2a}{\sigma^2}\frac{y'}{y} + 1 = 0$$

NB: $y(t) = \exp\left\{\frac{\sigma^2}{2}\int_t^T B(s,T)\,ds\right\}$

$\Rightarrow B(s,T) = -\dfrac{2}{\sigma^2}\dfrac{y'(t)}{y(t)}$

$$\Rightarrow B(\tau) = -\frac{2}{\sigma^2}\,\frac{x_2 \cancel{C_2}\left[-e \quad + e \quad\right]}{\cancel{C_2}\left[-\frac{x_2}{x_1}e^{x_1\tau} + e^{x_2\tau}\right]} = -\frac{2}{\sigma^2}\,\frac{x_2\left(e \quad - e \quad\right)}{\frac{e^{x_2\tau}x_1 - x_2 e^{x_1\tau}}{x_1}}$$

$$= -\frac{2}{\sigma^2}\,\frac{x_1 x_2\left(e^{x_2\tau} - e^{x_1\tau}\right)}{x_1 e^{x_2\tau} - x_2 e^{x_1\tau}} = -\frac{2}{\sigma^2}\,\frac{x_1 x_2 \cancel{e^{x_1\tau}}\left(e^{(x_2-x_1)\tau} - 1\right)}{\cancel{e^{x_1\tau}}\left(x_1 e^{(x_2-x_1)\tau} - x_2\right)}$$

$$\Rightarrow B(\tau) = -\frac{2}{\sigma^2}\,\frac{x_1 x_2\left(e^{(x_2-x_1)\tau} - 1\right)}{\left(x_1 e^{(x_2-x_1)\tau} - x_2\right)}$$

D'altra parte:

$$\Rightarrow x_1 \cdot x_2 = \frac{(a+h)(a-h)}{4} = \frac{1}{4}\left(a^2 - h^2\right)$$

$$x_2 - x_1 = h$$

$$\Rightarrow B(\tau) = -\frac{2}{\sigma^2}\,\frac{\frac{1}{4}\left(a^2-h^2\right)\left(e^{h\tau}-1\right)}{\frac{(a+h)}{2}e^{h\tau} - \frac{(a-h)}{2}} = -\frac{2}{2\sigma^2}\,\frac{\frac{1}{\cancel{2}}\left(a^2-h^2\right)\left(e^{h\tau}-1\right)}{\frac{(a+h)e^{h\tau}-(a-h)}{\cancel{2}}}$$

$$= -\frac{2}{2\sigma^2}\,\frac{-\frac{\left(h^2-a^2\right)}{\cancel{2}}\left(e^{h\tau}-1\right)}{\frac{(a+h)e^{h\tau}-(a+h-h-h)}{\cancel{2}}}$$

$$= +\frac{2}{\cancel{2\sigma^2}}\,\frac{\cancel{+2\sigma^2}\left(e^{h\tau}-1\right)}{(a+h)e^{h\tau}-(a+h)+2h}$$

$$\Rightarrow B(t,T) = \frac{2\left(e^{h(T-t)}-1\right)}{2h+(a+h)\left(e^{h(T-t)}-1\right)}$$

$$\Rightarrow A(t,T) = -\int_t^T a\gamma\, B(s,T)\,ds = -\int_t^T \frac{a\gamma\, 2\left[e^{h(T-s)}-1\right]}{2h+(a+h)\left(e^{h(T-s)}-1\right)}\,ds$$

$$= -\frac{2a\gamma}{\sigma^2}\ln\left[\frac{2h\, e^{\frac{a+h}{2}(T-t)}}{2h+(a+h)\left(e^{h(T-t)}-1\right)}\right]$$

$\boxed{\text{ESERCIZIO}}$

$$\Rightarrow -2y'' + 2ay' + \sigma^2 y = 0$$

$$\Rightarrow \underbrace{2y'' - 2ay' - \sigma^2 y = 0}_{\text{ODE del secondo ordine, lineare}} \quad (\text{ODE-L})$$

$\Rightarrow y = c_1 y_1 + c_2 y_2$, dove y_1, y_2 sono sol. particolari della eq. diff.

$\Rightarrow$ In particolare: $y_i = e^{x_i \tau}$, $\forall i = 1, 2$

$\Rightarrow y = c_1 e^{x_1 \tau} + c_2 e^{x_2 \tau}$

$$\Rightarrow y' = x e^{x\tau} = xy; \quad y'' = x^2 e^{x\tau} = x^2 y$$

$$\overset{(\text{ODE-L})}{\Rightarrow} 2x^2 \cancel{y} - 2ax\cancel{y} - \sigma^2 \cancel{y} = 0$$

$$\Rightarrow \underbrace{2x^2 - 2ax - \sigma^2 = 0}_{\text{Eq. caratteristica}}$$

$$\Rightarrow x = \frac{a \pm \sqrt{a^2 + 2\sigma^2}}{2} = \frac{a \pm h}{2},$$

avendo posto $h := \sqrt{a^2 + 2\sigma^2}$

$$\Rightarrow h^2 = a^2 + 2\sigma^2$$

$$\Rightarrow 2\sigma^2 = h^2 - a^2$$

$$\overset{(*)}{\Rightarrow} y = c_1 e^{x_1 \tau} + c_2 e^{x_2 \tau}, \quad y' = c_1 x_1 e^{x_1 \tau} + c_2 x_2 e^{x_2 \tau}$$

NB: c_1, c_2 costanti da determinare con le cond. al contorno

$$\Rightarrow B(\tau) = -\frac{2}{\sigma^2} \frac{c_1 x_1 e^{x_1 \tau} + c_2 x_2 e^{x_2 \tau}}{c_1 e^{x_1 \tau} + c_2 e^{x_2 \tau}} \quad (**)$$

$$B(0) = 0 \Rightarrow -\frac{2}{\sigma^2} \frac{c_1 x_1 e^{x_1 0} + c_2 x_2 e^{x_2 0}}{c_1 e^{x_1 0} + c_2 e^{x_2 0}} = 0$$

$$\Rightarrow c_1 x_1 \cancel{e^{x_1 0}} + c_2 x_2 \cancel{e^{x_2 0}} = 0 \Rightarrow c_1 = -c_2 \frac{x_2}{x_1}$$

$$\overset{(**)}{\Rightarrow} B(\tau) = -\frac{2}{\sigma^2} \frac{-c_2 \frac{x_2}{\cancel{x_1}} e^{x_1 \tau} \cancel{x_1} + c_2 x_2 e^{x_2 \tau}}{-c_2 \frac{x_2}{x_1} e^{x_1 \tau} + c_2 e^{x_2 \tau}}$$