Time Series Analysis

Prof. Lea Petrella

MEMOTEF Department
Sapienza, University of Rome, Italy

Class 13

ARCH models

- The most important element to be analyzed in financial data is the volatility.
- This is done by modelling and forecasting the conditional volatility by means of past data.
- A simple way to capture and forecast volatility is by means of *ARCH* models.
- An $ARCH(1)$ is defined as

$$r_t = \epsilon_t$$

where

$$\epsilon_t | \mathcal{F}_{t-1} \sim N(0, \sigma_t^2)$$

and

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2.$$

- Here $\mathcal{F}_{t-1}$ represents the information set at time $t-1$.
- Suppose $\omega > 0$ and $0 < \alpha < 1$ in order to have a stationary process and positive variance.

We show that r_t is a WN:

- The unconditional expected value is null:

$$\mathbb{E}(r_t) = 0$$

.

- This is obtained by using the law of total expectation:

$$\mathbb{E}(r_t) = \mathbb{E}(\mathbb{E}(r_t|\mathcal{F}_{t-1})) = 0$$

- The unconditional variance is constant

$$\begin{aligned}\sigma^2 =& \mathbb{E}(r_t^2) = Var(\epsilon_t) = \mathbb{E}(\epsilon_t^2) = \mathbb{E}(\mathbb{E}(\epsilon_t^2|\mathcal{F}_{t-1})) = \\ =& \mathbb{E}(\sigma_t^2) = \mathbb{E}(\omega + \alpha\epsilon_{t-1}^2) = \omega + \alpha\mathbb{E}(\epsilon_{t-1}^2) = \omega + \alpha\sigma^2 = \frac{\omega}{1-\alpha}.\end{aligned}$$

- Correlation is zero

$$\begin{aligned}Cov(r_t, r_{t-h}) =& \mathbb{E}(\epsilon_t\epsilon_{t-h}) = \mathbb{E}(\mathbb{E}(\epsilon_t\epsilon_{t-h}|\mathcal{F}_{t-1})) \\ =& \mathbb{E}(\epsilon_{t-h}\mathbb{E}(\epsilon_t|\mathcal{F}_{t-1})) = 0.\end{aligned}$$

- Since the covariance is zero, r_t cannot be predicted using $\mathcal{F}_{t-1}$. Notice, that this does not imply r_t independent. It can be shown that $Cov(r_t^2, r_{t-h}^2) \neq 0$.
- Even under Gaussianity, r_t is leptokurtic. To study tail behaviour we need to rely on the fourth moment. Recall that for a Gaussian distribution

$$\mathbb{E}(\epsilon_t^4|\mathcal{F}_{t-1}) = 3[\mathbb{E}(\epsilon_t^2|\mathcal{F}_{t-1})]^2.$$

- It follows

$$\begin{aligned}
\mu_4 =& \mathbb{E}(r_t^4) = \mathbb{E}(\epsilon_t^4) = \mathbb{E}(\mathbb{E}(\epsilon_t^4|\mathcal{F}_{t-1})) = \mathbb{E}(3\mathbb{E}(\sigma_t^2|\mathcal{F}_{t-1})^2) = \\
=& 3\mathbb{E}(\omega + \alpha\epsilon_{t-1}^2)^2 = 3\mathbb{E}(\omega^2 + \alpha^2\epsilon_{t-1}^4 + 2\omega\alpha\epsilon_{t-1}^2) = \\
=& 3\omega^2 + 3\alpha^2\mathbb{E}(\epsilon_{t-1}^4) + 6\omega\alpha\mathbb{E}(\epsilon_{t-1}^2). \\
& \implies \mu_4 = 3\omega^2 + 3\alpha^2\mu_4 + 6\omega\alpha\frac{\omega}{1-\alpha}.
\end{aligned}$$

- It follows
$$\mu_4 = \frac{3\omega^3(1+\alpha)}{(1-\alpha)(1-3\alpha^2)}.$$
- Kurtosis can be computed as
$$Kurt = \frac{\mathbb{E}(\epsilon_t^4)}{\mathbb{E}(\epsilon_t^2)^2} = 3\frac{1-\alpha^2}{1-3\alpha^2} \geq 3.$$
- Skewness is
$$\mu_3 = \mathbb{E}(r_t^3) = \mathbb{E}(\epsilon_t^3) = \mathbb{E}(\mathbb{E}(\epsilon_t^3|\mathcal{F}_{t-1}) = 0$$
- It can be shown that each odd moment is zero.
- In the *ARCH* models high shocks have an big impact on the conditional variance. Thus, they are appropprieate to capture the volatility clustering.

- It can be shown that an $ARCH(1)$ process an $ARMA$ representation. Specifically it has an $AR(1)$ representation,

$$r_t^2 = \omega + \alpha r_{t-1}^2 + v_t.$$

- Here v_t is a WN such that

$$\mathbb{E}(v_t) = 0$$

$$Var(v_t) = \frac{2\omega(1+\alpha)}{(1-\alpha)(1-3\alpha^2)},$$

and

$$Cov(v_t, v_{t-h}) = 0.$$

$$\mathbb{E}(r_t^2) = \frac{\omega}{1-\alpha}.$$

$$Var(r_t^2) = \frac{2\omega^2}{(1-\alpha)^2(1-3\alpha^2)}.$$

$$\rho_{r^2}(h) = Corr(r_t^2, r_{t-h}^2) = \alpha^h.$$

- That decays to zero according to the values of α.

- In general, an $ARCH(p)$ can be written as

$$r_t = \epsilon_t$$

where

$$\epsilon_t | \mathcal{F}_{t-1} \sim N(0, \sigma_t^2)$$

$$\sigma_t^2 = \omega + \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2.$$

- Notice that $\omega > 0$ and $\alpha_j \geq 0 \; \forall j$ ensure positive conditional variance.
- If $\sum_{j=1}^{p} < 1$ then $ARCH(p)$ is said weakly stationary.

- The model assumes that positive and negative shocks have same effect on the volatility, since it depends on the previous squared shocks.
- Instead, it is well known that prices react in a different way to positive and negative shocks (EGARCH, TGARCH).
- Moreover, volatility clusters are such that it is often necessary to use the class of $ACRH(p)$ with a high order of p.
- *ARCH* models exceed in the forecast of volatility and it can be shown that they react slowly to high and isolated shocks.