Time Series Analysis

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Class 1

Aim

- The aim of Time Series Analysis is to analyze a real phenomenon observed over time and to forecast it by using the set of information available up to a certain point in time t.
- Possible applications to:
  - micro and macro economics (GDP, TFP...)
  - business planning
  - production planning
  - production control and optimization of industrial processes
  - geographic and demographic surveys
  - financial markets forecasts
  - ...

- In order to be able to study time series models, recall the notion of stochastic process: a stochastic process is an ordered sequence of random variables indexed with t (time).
- More formally,

$$\{X_t(\omega), t \in T, \omega \in \Omega\},$$

where

  - T is a discrete or continuous set that orders the sequence of random variables. In this course we consider $T = \mathcal{N}$, that is only discrete time processes.
  - Ω is the sample space.

- A time series $\{x_t, t = 1, \ldots, n\}$ is one of the possible finite realizations of a stochastic process.
- Thus, it represents an ordered sequence of real numbers measuring the evolution over time (the variable t) of a phenomenon X_t.

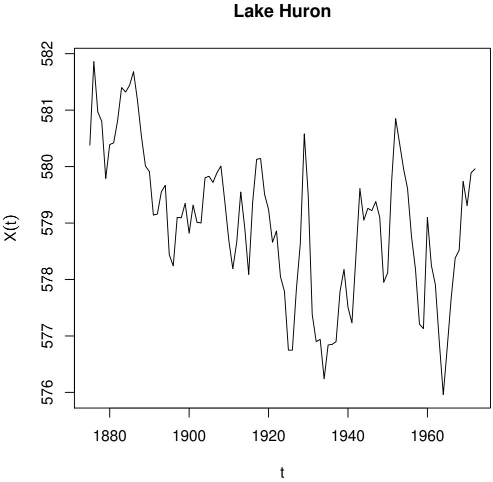

Figure: Annual measurements of the level, in feet, of Lake Huron 1875-1972.

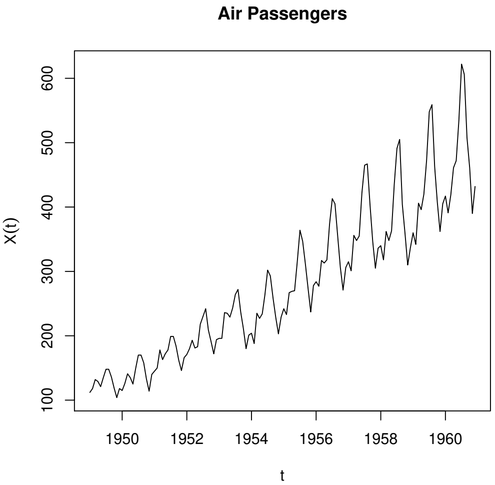

Figure: Monthly totals of international airline passengers, 1949 to 1960.

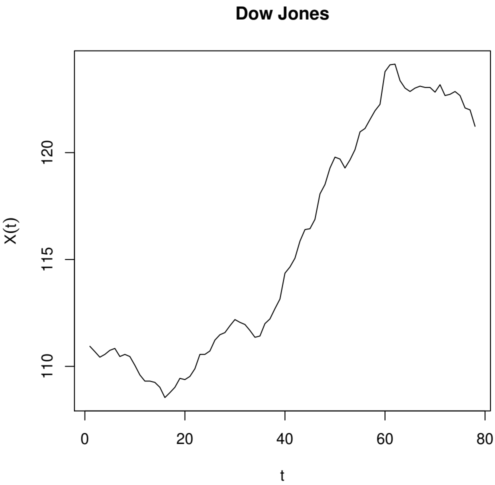

Figure: Dow Jones index time series, 28 August 1972 to 18 December 1972.

- Attention. A time series is a set of samples related to different random variables:
  - x_1 is the realization of the random variable $X_1(\omega)$,
  - x_n is the realization of the random variable $X_n(\omega)$.
- On the opposite, in classical inference the set $\{x_1, \ldots, x_n\}$ is sampled only from one random variable $X(\omega)$ and the observations are independent identically distributed (iid) as $X(\omega)$. Therefore, in classical inference the sample contains n different information for one random variable.
- In time series instead, the sample contains n different information for n random variables.
- Think about balls extractions from urns: for time series, it is as if we had n different urns and extract a ball from each urn. For the classical inference case, there is only one urn and n balls are extracted from the same urn.

- The expected value of X_t:

$$E(X_t) = \mu_t.$$

- The variance X_t:

$$Var(X_t) = \sigma_t^2.$$

- The covariance of X_r and X_s:

$$\gamma(r, s) = Cov(X_r, X_s) = E[(X_r - \mu_r)(X_s - \mu_s)]$$

more specifically, this measure is defined as autocovariance.

- In general, these moments are used to make inference. However, in time series analysis we are dealing with one single trajectory (the time series observed, and thus one observation for each random variable), and this does not allow make a valid inference under a statistical point of view.
- Therefore, two restrictions on the underlying process's characteristics are imposed:

1. restrictions regarding the heterogeneity of the process. This restriction is set in order to avoid excessive variability in the characteristics of X_t and thus it requires a certain homogeneity in its probabilistic structure (stationarity).
2. restrictions regarding the process's memory. This restriction is placed to avoid strong dependence in events distant in time (ergodicity).

- Ergodicity allows to use information in the time series to estimate the moments of the underlying process. However, In this course ergodicity won't be covered in detail.
- Example. A stationary process is said to be ergodic in mean if

$$\bar{X} = \frac{1}{n} \sum_{t=1}^{n} X_t \xrightarrow{p} \mu.$$

- A condition that must be satisfied for ergodicity is that the autocovariance function tends to zero fast (the process should not have long memory, i.e., by time t it should not be influenced by events distant in time).

- The class of stationary processes has received great attention both in theory and in applications.
- Such processes embed a long term equilibrium (e.g., with respect to their mean level).
- In this course we will study stationary processes, in particular we will show how stationary phenomena can be modelled. For non stationary processes we will make use of some manipulations to turn them into stationary ones.

Descriptive statistics

- Given that the time series is a sequence of observations in different points in times, the first necessary analysis is a graphical one.
- We plot the collected observations in the Cartesian axis with respect to the time in which they have been captured.
- The points in the axis have coordinates (t, x_t).

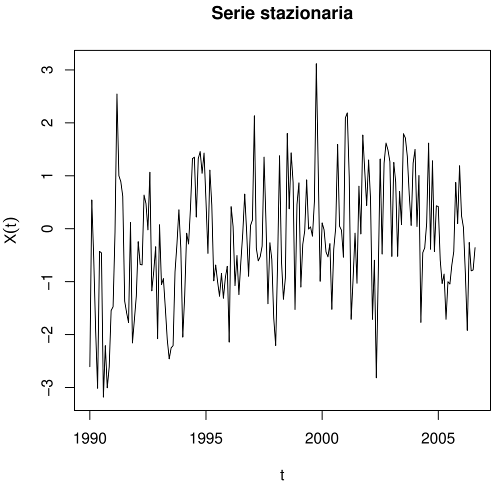

Figure: Example of an AR(1) time series with parameter 0.5.

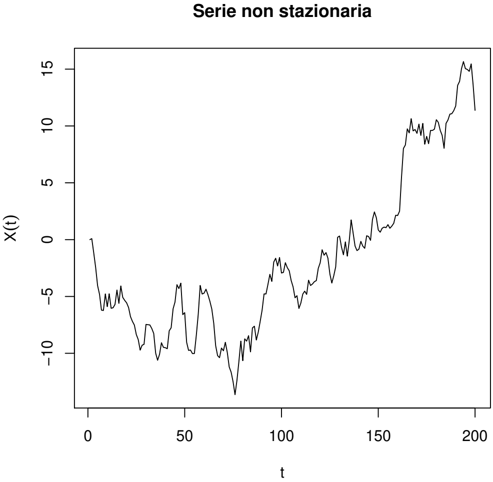

Figure: Example of non stationary time series.

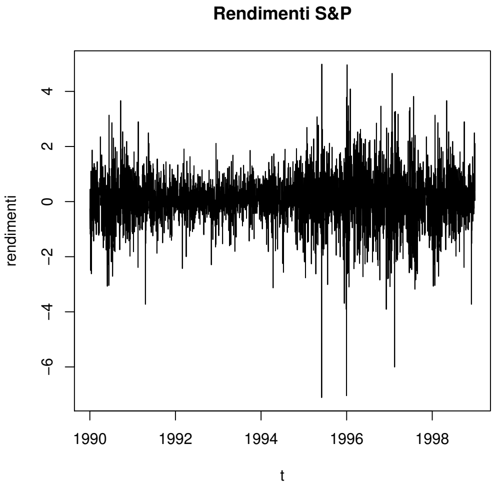

Figure: Example of non stationary time series.

Trend, seasonal component, accidental component

- It is not unusual to think of a time series as the outcome of several components (that may turn the process to be non stationary).
- The time series plot shows :

1. A trend cycle T_t explaining the long term path for the analyzed phenomenon, both in terms of regular evolution (trend, determined by structural evolution of the economic system) and seesaw (cycle, usually representing increasing or decreasing fluctuations of the phenomenon, generally connected with expansion or contraction phases of the entire economic system). The trend and the cycle components are jointly called trend.

Trend, seasonal component, accidental component

2. A seasonality S_t texplaining the periodicity component, i.e., detecting a feature (the same or very similar) at different fixed distances in time (e.g., in monthly series each 12 months, in quarterly series each 4 months, in daily series each 7 days...). Such effects are due to e.g., climate (seasons), convention (festivity that influences sales)....

   It is an external component that might not be included in a long term analysis, but can be of interest in short-mid term studies.

   In time series analysis, seasonality estimation is a delicate problem, since it can covere the real trajectory of the time series.

Trend, seasonal component, accidental component

3. An error ϵ_t representing the erratic component based on inexplicable shocks that influence fluctuations mostly in the short term. It is not systematic, but it is totally random. This component is usually seen as a stationary stochastic process whose determined as residual of the other components.

- Therefore, a time series can be written as an additive form of this kind:

$$X_t = T_t + S_t + \epsilon_t,$$

or in multiplicative form:

$$X_t = T_t \times S_t \times \epsilon_t$$

- It is always possible to transform a multiplicative form into an additive one by turning the series

$$X_t = T_t \times S_t \times \epsilon_t$$

into the log series

$$\log(X_t) = \log(T_t) + \log(S_t) + \log(\epsilon_t).$$

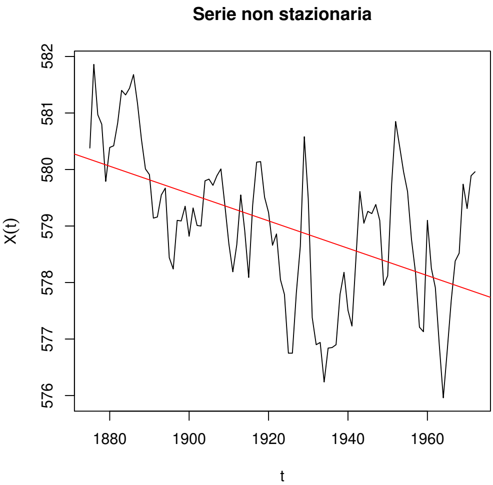

Figure: Example of non stationary time series whose non-stationarity is due to a trend: the line $x = 625.5549 - 0.0242t$.

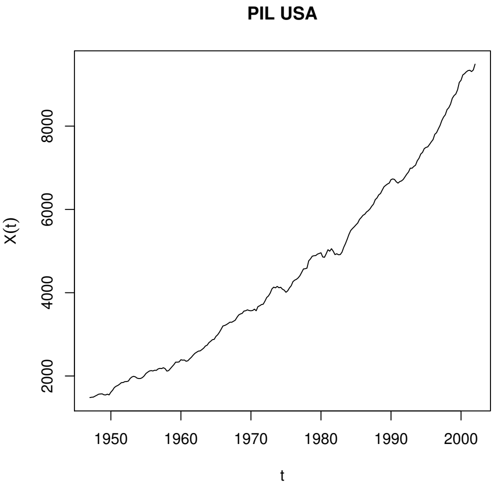

Figure: Example of non stationary time series, quarterly data from US GDP.

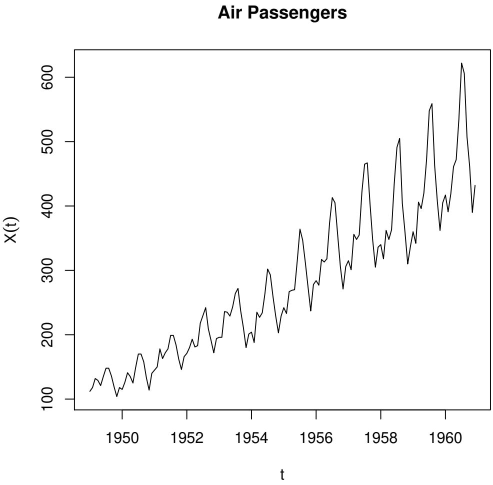

Figure: Airline passengers 1949 to 1960: non stationary time series with seasonality and increase in volatility.

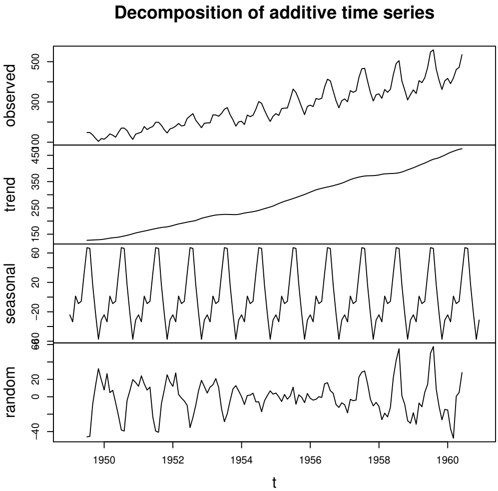

Figure: Identification of the components in Airline passengers time series.

- In order to obtain a stationaryprocess, it is often necessary to remove its trend and seasonal components. This is done by means of data manipulations:
- A reduction in the amplitude of linear fluctuations in the series can be easily obtained by transforming the data $\{x_1, \ldots, x_n\}$ into $\{\log x_1, \ldots, \log x_n\}$.

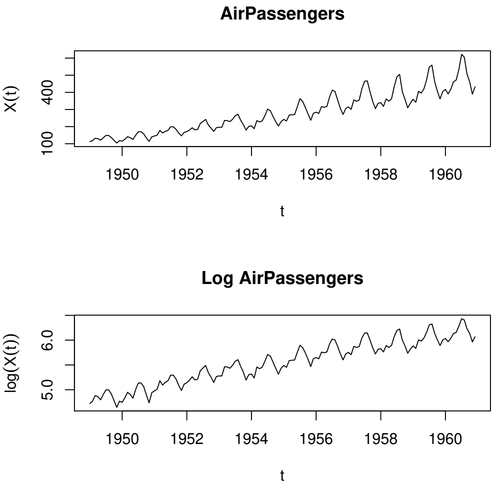

Figure: The use of logs to reduce fluctuations.

- Several ways to eliminate trend and/or seasonality. One idea is to perform estimation of such components and get rid of their estimated values in the original time series: by subtracting the estimates if the components are additive, or by dividing the estimates if the components are multiplicative.
- Different methodologies to estimate the components: least squares, moving average, exponential smoothing, splines...(we will not go through all the methodologies).

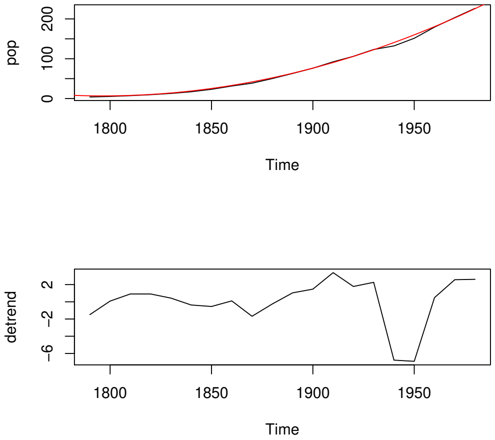

Figure: De-trending US population from 1790 to 1980. Trend estimate $y = 2.098e + 04 - 2.335e + 01t + 6.499e - 03t^2$.

- A polynomial trend of grade k can be removed by using the differences of order k of series, $\nabla^k x_t$.
- For instance, a linear trend in the series $\{x_t, t = 1, \dots, n\}$ can be removed with $\left\{y_t : y_t = x_t - x_{t-1} = \nabla^1 x_t, t = 1, \dots, n\right\}$.
- Example. Consider a series with linear trend $T_t = c_0 + c_1 t$, by applying the difference operator of order 1 we get: $\nabla_1 T_t = T_t - T_{t-1} = c_0 + c_1 t - c_0 - c_1(t-1) = c_1$. Iterating, the difference operator of order 2 gives: $\nabla^2 x_t = \nabla(\nabla x_t) = (x_t - x_{t-1}) - (x_{t-1} - x_{t-2}) = x_t - 2x_{t-1} + x_{t-2}$ so that a polynomial trend of grade 2 can be removed.

- An exponential trend component, $x_t = \beta_0 exp \beta_1 t + \beta_2 t^2$, can be removed by taking logs and then taking differences of the series as follows: $\nabla^2 \log(x_t)$.
- Attention. When considering differences of the series, the variance of residuals changes. In particular, it increasing in the order of the differences:
- $k = 0$; $Var(x_t) = \sigma^2$,
- $k = 1$; $Var(\nabla x_t) = Var(\nabla \epsilon_t) = Var(\epsilon_t) + Var(\epsilon_{t-1}) = 2\sigma^2$,
- $k = 2$;
  $Var(\nabla^2 x_t) = Var(\nabla^2 \epsilon_t) = Var(\epsilon_t) + 4Var(\epsilon_{t-1}) + Var(\epsilon_{t-2}) = 6\sigma^2$,
- $k = r$; $Var(\nabla^r x_t) = C_{2r,r}\sigma^2$.

- The seasonal component can be removed by considering seasonal differences of lag d for a seasonality of period d, i.e., $\nabla_d x_t = x_t - x_{t-d}$. For monthly seasonality: $\nabla_{12} x_t = x_t - x_{t-12}$.
- When both trend and seasonal component are detected then the following transformation can be applied: $\nabla \nabla_d = \nabla_d \nabla$.

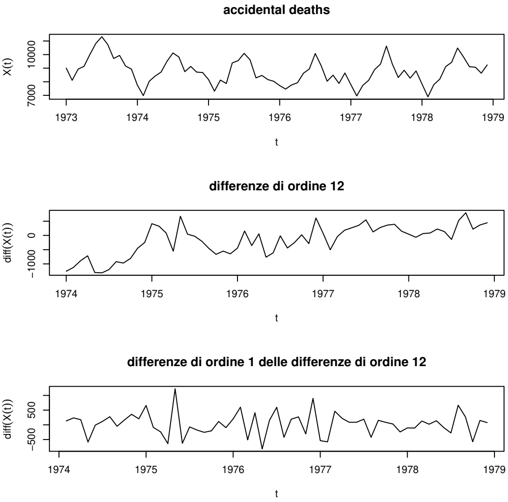

Figure: Difference in the series of mortal accidents in the USA from 1973 to 1977, in order to remove bothe trend and seasonal component.

Why removing seasonality ?

- Suppose that the average $CO2$ emissions in Rome has been higher in November than in October.
- Can you really conclude that pollution has increased ?
- That increase could be due to seasonality events, such as the increasing use of cars or heating (due to cold whether)...
- To remove seasonality allows to account for such component.
- Seasonality is often part of the time series and not very interesting but can be "big" enough to hide more important features.