Time Series Analysis

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Class 4

MA(1): Moving Average of first order

- The value of the random variable X_t (at time t) depends on the combination of two shocks (uncorrelated variables with zero mean and constant variance), the shock at time t and that related to a previous time $t-1$ with a coefficient θ:

$$X_t = \theta\epsilon_{t-1} + \epsilon_t$$

$$\epsilon_t \sim WN(0, \sigma^2)$$

$$(X_t = \mu + \theta\epsilon_{t-1} + \epsilon_t)$$

- Such process is considered as a result of the sum of past and present time random shocks.
- The *MA* models can be used to study phenomena with high irregularities characterized by a serial autocorrelation that vanishes after few lags.

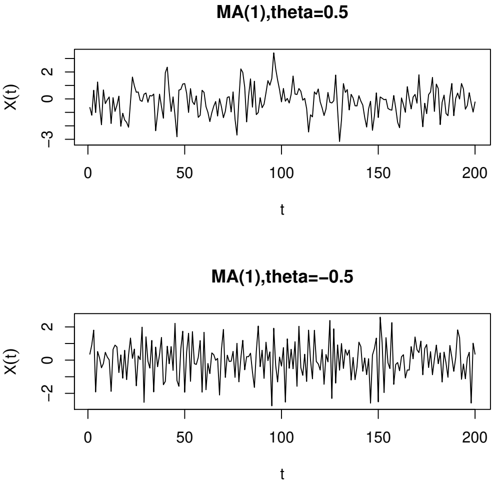


Figure: Path of two time series MA(1), zero mean where $\theta = 0.5$ and $\theta = -0.5$ and $\epsilon_t \sim WN(0, 1)$..

- The MA(1) process is always stationary:

$$\mathbb{E}(X_t) = \mathbb{E}(\theta\epsilon_{t-1} + \epsilon_t) = \theta\mathbb{E}(\epsilon_{t-1}) + \mathbb{E}(\epsilon_t) = 0$$

$$\mathbb{V}ar(X_t) = \gamma(0) = \mathbb{E}(\theta\epsilon_{t-1} + \epsilon_t)^2 =$$

$$= \mathbb{E}\left(\theta^2\epsilon_{t-1}^2 + \epsilon_t^2 + 2\theta\epsilon_t\epsilon_{t-1}\right) =$$

$$= \theta^2\sigma^2 + \sigma^2 + 0 = (1+\theta^2)\sigma^2$$

$$(\mathbb{E}(X_t) = \mathbb{E}(\mu + \theta\epsilon_{t-1} + \epsilon_t) = \mu + \theta\mathbb{E}(\epsilon_{t-1}) + \mathbb{E}(\epsilon_t) = \mu)$$

- The ACF a lag 1 is:

$$\gamma(1) = \mathbb{E}\left(X_t X_{t-1}\right) = \mathbb{E}\left[\left(\theta\epsilon_{t-1} + \epsilon_t\right)\left(\theta\epsilon_{t-2} + \epsilon_{t-1}\right)\right] =$$

$$\theta^2\mathbb{E}\left(\epsilon_{t-1}\epsilon_{t-2}\right) + \theta\mathbb{E}\left(\epsilon_{t-1}^2\right) + \theta\mathbb{E}\left(\epsilon_t\epsilon_{t-2}\right) + \mathbb{E}\left(\epsilon_t\epsilon_{t-1}\right) =$$

$$= \theta^2 \times 0 + \theta \times \sigma^2 + \theta \times 0 + 0 = \theta\sigma^2$$

$$\gamma(h) = 0 \quad \textit{for} \;\; h \geq 2.$$

- Autocovariances vanish since the ϵ_t are uncorrelated.
- The memory of a process $\{X_t\}$ of the kind $MA(1)$ lasts for only one period (lag).
- Stationarity: there is no dependence of time.

Sample ACF

- Compute the autocorrelation (see how the ACF looks like):

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2} \quad \Rightarrow \quad -0.5 \leq \rho(1) \leq 0.5$$

$$\rho(h) = 0 \quad \forall h \geq 2.$$

- The maximum of $\rho(1)$ is obtained when $\theta = 1$ and the minimum when $\theta = -1$. In those values the ACF is, respectively, 0.5 e -0.5.
- The theoretical ACF of an MA(1) can be plotted as a function of h for different values of θ.

Stationary property

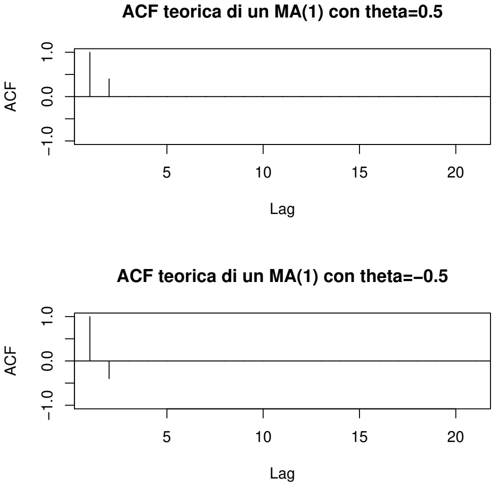
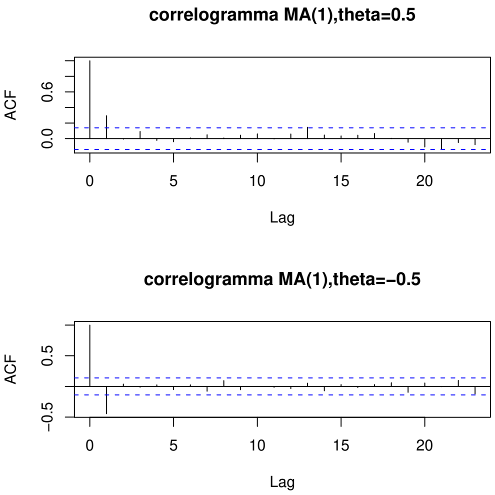


Figure: Theoretical ACF of two MA(1) processes.

- For simulated series correlograms are the analogue of ACF.



- PACF is given by:

$$\phi_{hh} = \frac{-(-\theta)^h}{1 + \theta^2 + \ldots + \theta^{2h}},$$

that is,

$$\phi_{11} = \frac{\theta}{1 + \theta^2}, \quad \phi_{22} = \frac{-(-\theta)^2}{1 + \theta^2 + \theta^4}, \ldots$$

- In contrast with the ACF that decays at the first lag, PACF of an MA(1) decays with alternating signs if $\theta > 1$ and vanishes at an exponential rate after the first lag if $\theta < 1$.
- PACF goes to zero slowly after the first lag.

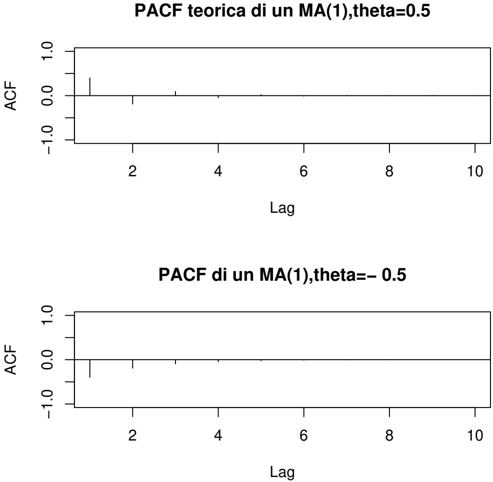
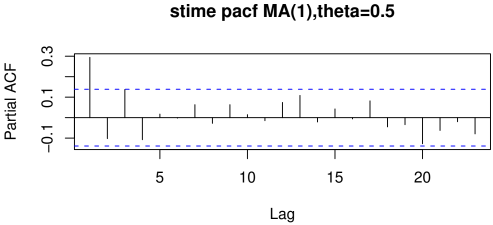
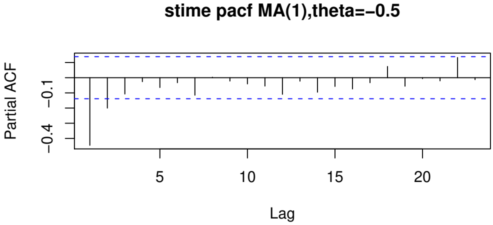


Figure: Theoretical PACF of two MA(1) processes.

- For the simulated series estimated PACF are:





The identification issue: Invertibility

- Consider the following $MA(1)$ processes:

$$X_t = \theta\epsilon_{t-1} + \epsilon_t$$
$$Y_t = \theta^*\epsilon_{t-1} + \epsilon_t,$$

where

$$\theta^* = \frac{1}{\theta}.$$

Then,

$$\rho_X(1) = \frac{\theta}{1+\theta^2} \qquad \rho_Y(1) = \frac{\theta^*}{1+\theta^{*2}}$$

- The ACF is the same:

$$\rho_Y(1) = \frac{1/\theta}{1+1/\theta^2} = \frac{\theta}{1+\theta^2} = \rho_X(1)$$

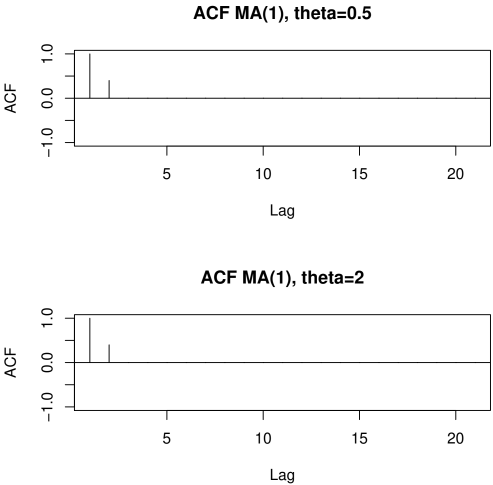


Figure: MA(0.5) and MA(2.0) show same ACF.

- It has been shown that two $MA(1)$ processes (of the same order) with parameters one the reciprocal of the other, have same autocorrelations.
- Therefore, it is not possible to identify the process underlying the observed time series only by looking at the ACF.
- This issue concerns the entire class of MA proceses.
- In order to avoid such problem we consider invertible MA, that is processes that satisfy $|\theta| < 1$.
- It can be shown that if $|\theta| < 1$, the process $X_t = \theta\epsilon_{t-1} + \epsilon_t$ has a representation of the kind: $X_t = h(X_{t-1}, X_{t-2}, \cdots) + \epsilon_t^*$.
- The Invertibility condition is independent of stationarity condition.

- The lag operator allows to write the MA(1) as:

$$X_t = \theta\epsilon_{t-1} + \epsilon_t = (1 + \theta B)\epsilon_t = \Theta(B)\epsilon_t$$

- It can be shown that such process is invertibile if the characteristic equation $\Theta(B) = 0$ admits a (unique) root in B, $|B| > 1$, i.e.,

$$|B| > 1 \Leftrightarrow |\theta^{-1}| > 1 \Leftrightarrow |\theta| < 1.$$

- If $|\theta| < 1$ then

$$(1+\theta B)^{-1} = (1-(-\theta B))^{-1} = \sum_{j=0}^{\infty}(-\theta)^j B^j.$$

Then,

$$\epsilon_t = (1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \ldots)X_t$$

so

$$X_t = \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 X_{t-3} + \ldots + \epsilon_t.$$

Impulse function

- The impulse function measures the effect of a unit variation of the random variable ϵ on the X through time. Specifically, this function assesses the persistence of the random shock.
- For an $MA(1)$ process it is given by:

$$\frac{\partial X_t}{\partial \epsilon_t} = 1,$$

$$\frac{\partial X_{t+1}}{\partial \epsilon_t} = \theta,$$

$$\frac{\partial X_{t+j}}{\partial \epsilon_t} = 0 \quad \forall j > 1.$$

- The shock has effect on one period ahead only and its intensity depends on the value of the parameter θ.

- The $MA(1)$ model is very parsimonious because it contains only one unknown parameter but it is not flexible enough as the information at time t become useless starting at time $t + 2$.
- Moreover, ACF only lasts one lag and the maximum value it reaches is $|0.5|$.
- We will study how to make this model more flexible by increasing the memory of the process.