Time Series Analysis

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Class 5

MA(2): Moving Average of second order

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$$
$$\epsilon_t \sim WN(0, \sigma^2)$$

$$(X_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2})$$

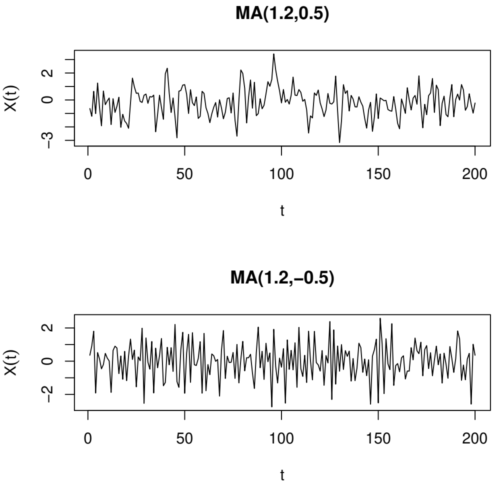

Figure: Time series simulated from two MA(2).

- Let's evaluate the stationarity and compute the moments of order two:

$$\mathbb{E}\left(X_t\right) = 0$$

$$Var\left(X_t\right) = \left(1 + \theta_1^2 + \theta_2^2\right)\sigma^2$$

$$\gamma(1) = Cov(X_t, X_{t-1}) =$$

$$= \mathbb{E}\left[\left(\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}\right)\left(\epsilon_{t-1} + \theta_1\epsilon_{t-2} + \theta_2\epsilon_{t-3}\right)\right] =$$

$$= \left(\theta_1 + \theta_1\theta_2\right)\sigma^2 = \theta_1\left(1 + \theta_2\right)\sigma^2$$

$$\gamma(2) = Cov(X_t, X_{t-2}) =$$

$$= \mathbb{E}\left[\left(\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}\right)\left(\epsilon_{t-2} + \theta_1\epsilon_{t-3} + \theta_2\epsilon_{t-4}\right)\right] = \theta_2\sigma^2$$

$$\gamma(h) = 0 \quad \forall h \geq 3.$$

- The ACF of the process is given by:

$$\rho(1) = \frac{\theta_1\left(1+\theta_2\right)}{\left(1+\theta_1^2+\theta_2^2\right)}$$

$$\rho(2) = \frac{\theta_2}{\left(1+\theta_1^2+\theta_2^2\right)}$$

$$\rho(h) = 0 \quad per \quad h \geq 3.$$

- An $MA(2)$ process is always stationary and has ACF different from zero up to order 2.

Stationary property

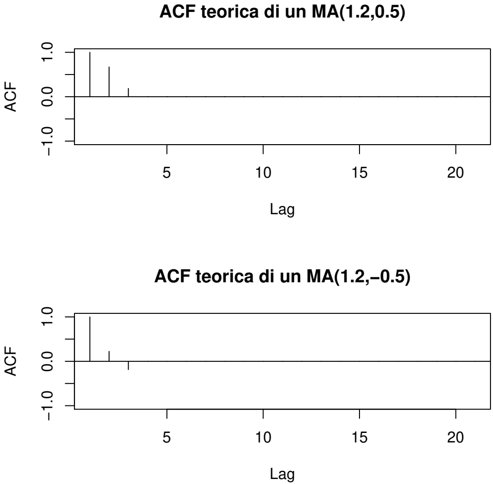

Figure: Theoretical ACF of two MA(2) processes.

- In order to assess the invertibility, consider the representation

$$X_t = \left(1 + \theta_1 B + \theta_2 B^2\right) \epsilon_t = \Theta(B)\epsilon_t$$

  that requires the two roots in B of the characteristic function associated with $\Theta(B) = 0$ to be in module greater than 1.
- This condition implies finding a triangular region of admissible solutions for the parameters in the $MA(2)$ model

- The characteristic polynomial has roots in B given by:

$$B_1 = \frac{-\theta_1 + \sqrt{\theta_1^2 - 4\theta_2}}{2\theta_2},$$

$$B_2 = \frac{-\theta_1 - \sqrt{\theta_1^2 - 4\theta_2}}{2\theta_2}.$$

- Imposing $|B_i| > 1$ means $1/|B_i| < 1$ for $i = 1, 2$, where,

$$\frac{1}{B_1} = \frac{-\theta_1 - \sqrt{\theta_1^2 - 4\theta_2}}{2},$$

$$\frac{1}{B_2} = \frac{-\theta_1 + \sqrt{\theta_1^2 - 4\theta_2}}{2}.$$

- We can write
$$\left|\frac{1}{B_1} \times \frac{1}{B_2}\right| = |\theta_2| < 1,$$
and
$$\left|\frac{1}{B_1} + \frac{1}{B_2}\right| = |\theta_1| < 2.$$
- Therefore, necessary conditions for the $MA(2)$ to be invertible are:
$$-1 < \theta_2 < 1$$
$$-2 < \theta_1 < 2.$$
- If the roots were real or complex we would consider the following conditons: $\theta_1^2 - 4\theta_2 > 0$ or $\theta_1^2 - 4\theta_2 < 0$.

- Simple algebra shows that the invertibility condition for an $MA(2)$ is given by the triangular region that satisfies:

$$-\theta_1 - \theta_2 < 1,$$

$$\theta_1 - \theta_2 < 1,$$

$$-1 < \theta_2 < 1.$$

- An $MA(2)$ process is always stationary but invertible only for an aprrropriate choice of the parameters.
- The PACF is not easy to compute. However, it goes to zero slowly, at different rates depending on the roots (real or complex) of the invertible process.

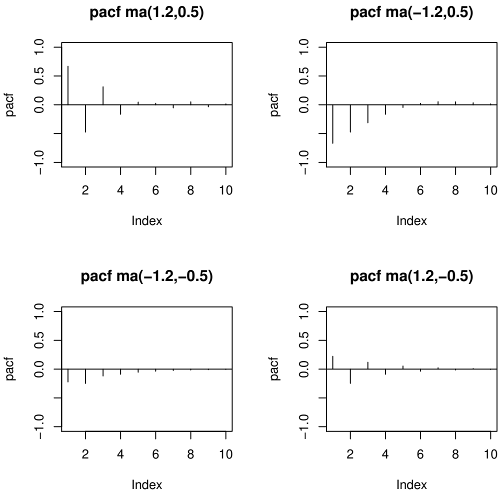

Figure: ACF and PACF of MA(2) processes.

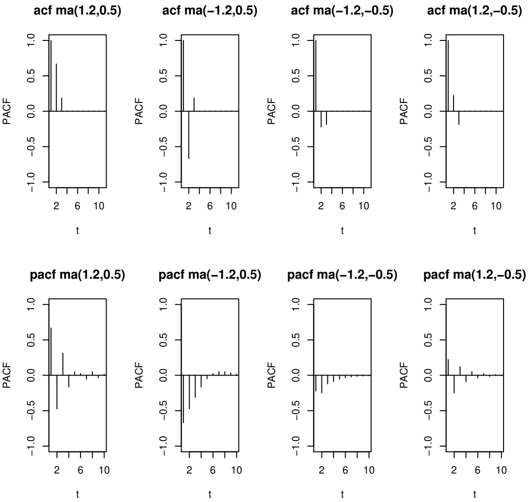

Figure: ACF and PACF of MA(2) processes.