Time Series Analysis

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Class 6

MA(q): Moving Average of order q

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots + \theta_q \epsilon_{t-q},$$
$$\epsilon_t \sim WN(0, \sigma^2),$$

$$\left(X_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots + \theta_q \epsilon_{t-q}\right).$$

- Let's evaluate the stationarity and compute the moments of order 2.

$$\mathbb{E}\left(X_t\right) = 0$$

$$Var\left(X_t\right) = \gamma(0) = \left(1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2\right) \sigma^2.$$

- For $h = 1, 2, \ldots, q$, the autocovariance function is

$$\gamma(h) = \mathbb{E}[(\epsilon_t + \theta_1 \epsilon_{t-1} + \ldots + \theta_h \epsilon_{t-h} + \theta_{h+1}\epsilon_{t-h-1} + \ldots + \theta_q \epsilon_{t-q}) \times$$

$$(\epsilon_{t-h} + \theta_1 \epsilon_{t-h-1} + \ldots + \theta_h \epsilon_{t-h-h} + \theta_{h+1}\epsilon_{t-h-1} + \theta_{h+2}\epsilon_{t-h-2}\theta_q \epsilon_{t-h-q})]$$
$$= \mathbb{E}\left(\theta_h \epsilon_{t-h}^2 + \theta_1\theta_{h+1}\epsilon_{t-h-1}^2 + \theta_2\theta_{h+2}\epsilon_{t-h-2}^2 + \cdots \theta_q\theta_{q-h}\epsilon_{t-h-q}^2\right).$$

So,

$$\gamma(h) = (\theta_h + \theta_{h+1}\theta_1 + \theta_{h+2}\theta_2 + \ldots + \theta_q\theta_{q-h})\,\sigma^2 \quad per\ h = 1, 2, \ldots, q$$

and

$$\gamma(h) = 0 \quad per\ \ h > q.$$

- An $MA(q)$ process is always stationary and has finite memory of order q.

- Example. For an $MA(2)$:

$$\gamma(0) = (1 + \theta_1^2 + \theta_2^2)\sigma^2$$

$$\gamma(1) = (\theta_1 + \theta_2\theta_1)\sigma^2 = \theta_1(1 + \theta_2)\sigma^2$$

$$\gamma(2) = (\theta_2)\sigma^2$$

$$\gamma(3) = \gamma(4) = \cdots = 0$$

- In order to assess the invertibility, condsider the representation:

$$X_t = \left(1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q\right)\epsilon_t = \Theta(B)\epsilon_t$$

  for which the invertibility conditions require the q roots in B of the characteristic function associated with $\Theta(B) = 0$ to be in module greater than 1.

MA(∞)

- Let's write the $MA(q)$ as

$$X_t = \sum_{j=0}^{q} \theta_j \epsilon_{t-j},$$

with $\theta_0 = 1$, and consider the resulting process imposing $q \to \infty$.

$$X_t = \sum_{j=0}^{\infty} \theta_j \epsilon_{t-j} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots.$$

- This process is said to be $MA(\infty)$, it can be stationary if

$$\sum_{j=0}^{\infty} \theta_j^2 < \infty,$$

i.e., if the sum of the squared sequence $\{\theta_j\}_{j=0}^{\infty}$ is finite.

- It may be convenient to impose a stronger condition,

$$\sum_{j=0}^{\infty} |\theta_j| < \infty.$$

- That is the sum of the module of the sequence $\{\theta_j\}_{j=0}^{\infty}$ is finite.
- An MA(∞) process is stationary if the aforementioned condition holds.

- Mean and second moments can be can be obtained as limit of those obtained for the $MA(q)$ by studying what happens when $q \to \infty$.
- Thus,

$$\mathbb{E}(X_t) = 0,$$

$$\gamma(0) = \sigma^2 \sum_{k=0}^{\infty} \theta_k^2,$$

$$\gamma(h) = \sigma^2 \sum_{k=0}^{\infty} \theta_k \theta_{k+h}.$$