Time Series Analysis

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Class 7

AR(1): Autoregressive of order 1

- Autoregressive models are used to explain phenomena whose present value can be derived by their past value plus a random shock.
- A better interpretation and diffusion of this class of models with respect to the Moving Average class is due to their similarities with the linear regression model.

$$X_t = \varphi X_{t-1} + \epsilon_t,$$

$$\epsilon_t \sim WN(0, \sigma^2),$$

$$\left(X_t = \delta + \varphi X_{t-1} + \epsilon_t\right).$$

AR(1): Autoregressive of order 1

- If $|\varphi| \geq 1$ then the process X_t would explode (to $\pm\infty$) because the shocks ϵ_t would accumulate and would not vanish in time.
- It is not surprising that when $|\varphi| \geq 1$ the Autoregressive process is not stationary.

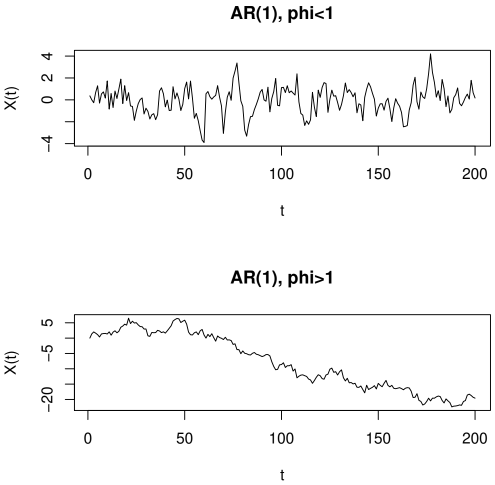


Figure: Stationary and non stationary $AR(1)$ time series.

- More formally, iterating substitutions of the process:

$$X_t = \varphi X_{t-1} + \epsilon_t = \varphi(\varphi X_{t-2} + \epsilon_{t-1}) + \epsilon_t = \varphi^2 X_{t-2} + \varphi\epsilon_{t-1} + \epsilon_t =$$

$$= \varphi^2(\varphi X_{t-3} + \epsilon_{t-2}) + \varphi\epsilon_{t-1} + \epsilon_t = \varphi^3 X_{t-3} + \varphi^2\epsilon_{t-2} + \varphi\epsilon_{t-1} + \epsilon_t =$$

$$\dots\dots\dots$$

$$= \epsilon_t + \varphi\epsilon_{t-1} + \varphi^2\epsilon_{t-2} + \dots \to MA(\infty),$$

- which is stationary if $|\varphi| < 1$.
- Taking the expected value and recalling that

$$\sum_{j=0}^{\infty} \varphi^j = \frac{1}{1-\varphi}$$

if $|\varphi| < 1$, then:

$$\mathbb{E}\left(\epsilon_t + \varphi\epsilon_{t-1} + \varphi^2\epsilon_{t-2} + \dots\right) = \frac{0}{1-\varphi} < \infty.$$

- For the variance we have:

$$Var(X_t) = \gamma(0) = \mathbb{E}(X_t - \mu)^2 = \mathbb{E}(X_t)^2 = E(\epsilon_t + \varphi\epsilon_{t-1} + \varphi^2\epsilon_{t-2} + \ldots\ldots)^2$$

$$= Var(\epsilon_t + \varphi\epsilon_{t-1} + \varphi^2\epsilon_{t-2} + \ldots\ldots) =$$

$$= (1 + \varphi^2 + \varphi^4 + \varphi^6 + \ldots)\sigma^2 = \sigma^2 \sum_{j=0}^{\infty} \varphi^{2j} = \frac{\sigma^2}{1 - \varphi^2},$$

if $|\varphi| < 1$

- The autocovaiance function is given by:

$$\gamma(h) = \mathbb{E}\left[\left(\epsilon_t + \varphi\epsilon_{t-1} + \varphi^2\epsilon_{t-2} + \ldots + \varphi^h\epsilon_{t-h} + \varphi^{h+1}\epsilon_{t-h-1} + \right.\right.$$

$$\left.\left.\varphi^{h+2}\epsilon_{t-h-2} + \ldots\right)\right] \times (\epsilon_{t-h} + \varphi\epsilon_{t-h-1} + \varphi^2\epsilon_{t-h-2} + \ldots)] =$$

$$= (\varphi^h + \varphi^{h+2} + \varphi^{h+4} + \ldots)\sigma^2 = \varphi^h(1 + \varphi^2 + \varphi^4 + \ldots)\sigma^2 =$$

$$= \sigma^2 \frac{\varphi^h}{1 - \varphi^2} = \varphi^h \gamma(0),$$

if $|\varphi| < 1$.

- The ACF is given by:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \varphi^h.$$

- When $\varphi > 0$, the ACF decays exponentially to zero.
- When $\varphi > 0$, the ACF decays exponentially to zero but with positive and negative fluctuations.

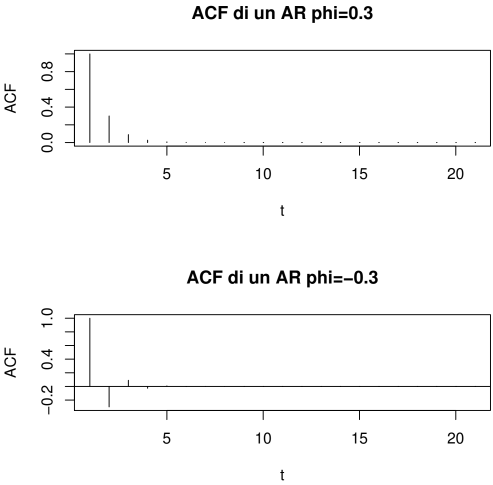


Figure: ACF of two AR(1) stationary processes.

1 The autocorrelations, $\rho(h)$ are the elements of a progressive geometric series that converges to zero as $|\varphi| < 1$,

$$\lim_{h\to\infty} \rho(h) = \lim_{h\to\infty} \varphi^h = 0 \quad if\ |\varphi| < 1.$$

2 The autocovariance function can be written recursively by using a first difference equation:

$$\gamma(h) = \varphi\gamma(h-1), \qquad h > 0.$$

3 The autocorrelation function can be written in a similar way:

$$\rho(h) = \varphi\rho(h-1).$$

4 The impulse response function is equal to:

$$\frac{\partial X_{t+j}}{\partial \epsilon_t} = \varphi^j,$$

that is, the ACF and in the long-run the effect of a shock vanishes if $|\varphi| < 1$.

5 The greater the parameter φ the greater the correlation with the past, the greater is the effect of a shock in time.

6 A stationary $AR(1)$ process can be written in terms of $MA(\infty)$.

- Results 1,2,3,4,5 allows to understand the stationarity condition, if $|\varphi| > 1$ those effects would amplify with time.

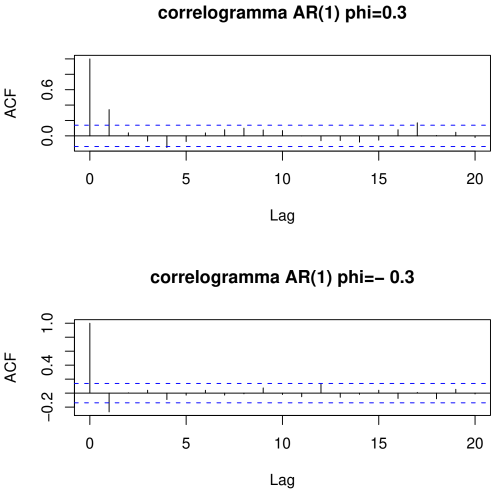


Figure: Correlogram of stationary AR(1) processes.

- An alternative way to obtain stationary conditions is to consider the representation of the $AR(1)$ in terms of B operator:

$$X_t = \varphi X_{t-1} + \epsilon_t$$

$$\Leftrightarrow$$

$$X_t - \varphi X_{t-1} = \epsilon_t$$

$$\Leftrightarrow$$

$$(1 - \varphi B)X_t = \epsilon_t$$

- This condition requires the roots in B of $(1 - \varphi B) = \Phi(B) = 0$ to lie outside the unit circle, that is, $|B| > 1 \Leftrightarrow |\varphi| < 1$.

- An alternative way to obtain the $MA(\infty)$ representation of an $AR(1)$ is to consider the B operator starting from: $(1-\varphi B)X_t = \epsilon_t$ and to derive

$$X_t = (1-\varphi B)^{-1}\epsilon_t = \frac{1}{(1-\varphi B)}\epsilon_t.$$

- For stationary process we can then write:

$$\frac{1}{(1-\varphi B)} = \sum_{i=0}^{\infty}(\varphi B)^i = 1 + (\varphi B) + (\varphi B)^2 + \ldots\ldots$$

- Thus,

$$X_t = \frac{1}{(1-\varphi B)} = \sum_{i=0}^{\infty}(\varphi B)^i \epsilon_t = \sum_{i=0}^{\infty}(\varphi)^i \epsilon_{t-i}.$$

- For PACF, recalling what we saw before, we have

$$\phi_{11} = \rho(1) = \varphi.$$

- For ϕ_{kk}, recalling the definition of $AR(1)$, we have

$$\phi_{kk} = 0 \quad \forall k \geq 2.$$

- For example. $\phi_{22} = 0$. If it was not zero, we could write:

$$X_t = \phi_{21} X_{t-1} + \phi_{22} X_{t-2} + e_t,$$

- However, since the process is autoregressive of first order, it must be $\phi_{22} = 0$.

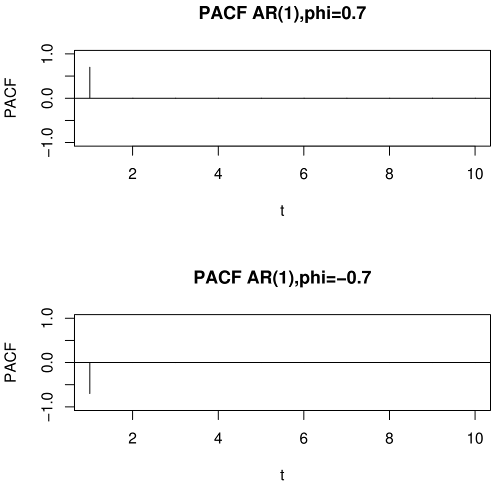


Figure: PACF of stationary AR(1) processes.

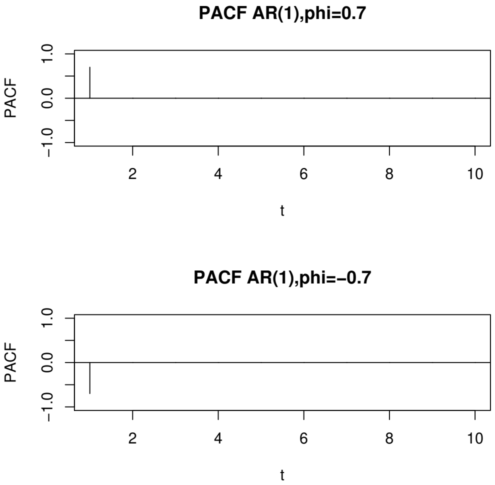


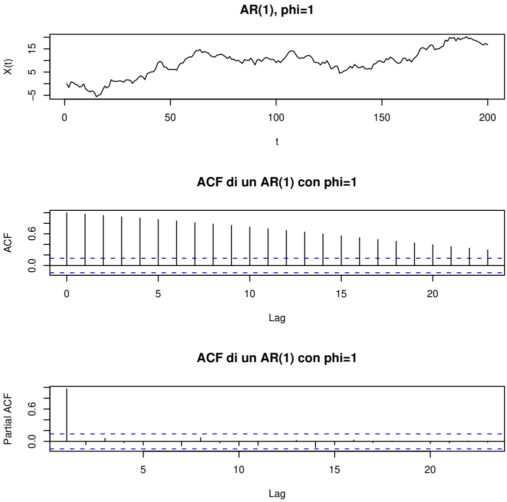
Figure: Estimated PACF of stationary AR(1) processes.

- Before studying $AR(2)$ processes, consider the case in which $\varphi = 1$, i.e., the process has Unit Root.
- in this case the model writes:

$$X_t = X_{t-1} + \epsilon_t,$$

$$\epsilon_t \sim WN(0, \sigma^2).$$

- It is a kind of random walk process.
- The time series and its correlogram may look like the following:


AR(1), phi=1
X(t)
−5
5
15
0
50
100
150
200
t
ACF di un AR(1) con phi=1
ACF
0.0
0.6
0
5
10
15
20
Lag
ACF di un AR(1) con phi=1
Partial ACF
0.0
0.6
5
10
15
20
Lag

- Applying the first differences we would get:

$$\nabla X_t = \epsilon_t.$$

- That is, a stationary process. In this case the process X_t is first order integrated, $X_t \sim I(1)$.
- It means that the process X_t needs to be differentiated once in order to be stationary.
- We already discussed that the random walk in not stationary. Moreover, it is possible to claim that a shock has permanent effect:

$$\frac{\partial X_{t+h}}{\epsilon_t} = 1 \quad \forall h > 0.$$

- If the $AR(1)$ process is of the kind ($X_t = \delta + \varphi X_{t-1} + \epsilon_t$), it can be observed that the process is stationary. Then, $\mathbb{E}(X_t) = \mu \ \forall t$ for which

$$\mathbb{E}(X_t) = \delta + \varphi \mathbb{E}(X_{t-1}) + \mathbb{E}(\epsilon_t),$$

$$\mu = \delta + \varphi \mu,$$

$$\mu = \frac{\delta}{(1 - \varphi)}.$$