Time Series Analysis

Prof. Lea Petrella

MEMOTEF Department
Sapienza, University of Rome, Italy

Class 8

AR(2): Autoregressive of order 2

$$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \epsilon_t$$
$$\epsilon_t \sim WN(0, \sigma^2)$$
$$(X_t = \delta + \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \epsilon_t) .$$

- AR(2) can be written as
$$(1 - \varphi_1 B - \varphi_2 B^2) X_t = \epsilon_t$$

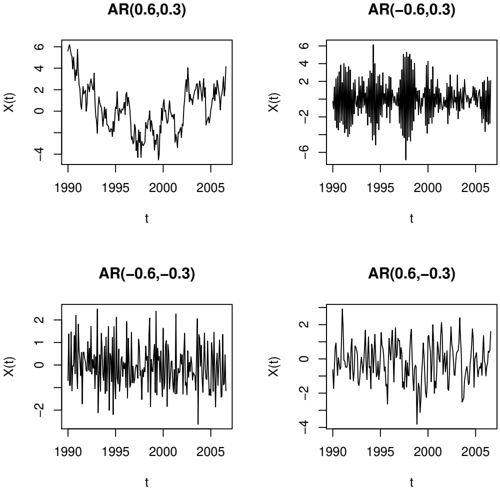


Figure: $AR(2)$ time series.

- The stationarity condition requires the roots of the characteristic equation $(1 - \varphi_1 B - \varphi_1 B^2) = \Phi(B) = 0$ to lie outside the unit circle.
- This implies φ_1 e φ_2 should be inside the triangular region

$$\varphi_1 + \varphi_2 < 1$$
$$\varphi_2 - \varphi_1 < 1$$
$$-1 < \varphi_2 < 1.$$

- Before computing the moments of the process let's have a look at is $MA(\infty)$ representation.
- Suppose the absolute value of $\frac{1}{\psi_1}$ and $\frac{1}{\psi_2}$ is greater than one, and let them be solutions of the characteristic equation

$$(1 - \varphi_1 B - \varphi_1 B^2) = \Phi(B) = 0,$$

- equivalently,

$$(1 - \varphi_1 B - \varphi_1 B^2) = (1 - \psi_1 B)(1 - \psi_2 B)$$

- Thus, the $AR(2)$ process can be written as:

$$(1 - \psi_1 B)(1 - \psi_2 B) X_t = \epsilon_t$$

$$\Leftrightarrow$$

$$X_t = \frac{1}{(1 - \psi_1 B)} \frac{1}{(1 - \psi_2 B)} \epsilon_t$$

- That is,

$$X_t = \Phi^{-1}(B)\epsilon_t$$

$$\Leftrightarrow$$

$$X_t = \sum_{i=0}^{\infty}(\psi_1 B)^i \sum_{j=0}^{\infty}(\psi_2 B)^j \epsilon_t = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(\psi_1)^i(\psi_2)^j \epsilon_{t-i-j}.$$

- From which we have

$$E(X_t) = 0.$$

- In order to obtain the second moments, multiply the equation $X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \epsilon_t$ of the $AR(2)$ model by X_{t-h} and consider the expected value:

$$\mathbb{E}(X_t X_{t-h}) = \varphi_1 \mathbb{E}(X_{t-1}X_{t-h}) + \varphi_2 \mathbb{E}(X_{t-2}X_{t-h}) + \mathbb{E}(\epsilon_t X_{t-h})$$

$$\gamma(h) = \varphi_1 \gamma(h-1) + \varphi_2 \gamma(h-2) \quad \textit{for} \;\; j = 1, 2, \ldots$$

- For $j = 0$ we have,

$$\gamma(0) = \varphi_1\gamma(1) + \varphi_2\gamma(2) + \sigma^2$$

  where we used the property $\gamma(h) = \gamma(-h)\ \forall h$.

- For the autocorrelation we have:

$$\rho(h) = \varphi_1\rho(h-1) + \varphi_2\rho(h-2) \quad \textit{for} \;\; h = 1, 2, \ldots$$

  that varies with h and determines a system of linear equation known as Yule-Walker equation.

- More specifically, we can observe that:

$$\gamma(0) = \frac{\sigma^2(1-\varphi_2)}{[(1-\varphi_2)^2 - \varphi_1^2](1+\varphi_2)}.$$

- When $h = 1$ and $h = 2$

$$\rho(1) = \varphi_1 + \varphi_2\rho(1)$$
$$\rho(2) = \varphi_1\rho(1) + \varphi_2$$

- From which

$$\rho(1) = \frac{\varphi_1}{1-\varphi_2}$$

$$\rho(2) = \frac{\varphi_1^2}{1-\varphi_2} + \varphi_2 = \frac{\varphi_1^2 + \varphi_2 - \varphi_2^2}{1-\varphi_2}$$

- It can be shown that the ACF decays
  - exponentially if the roots of the characteristic equation are real,
  - in sinusoidal mode if the roots are complex.
- Overall, the ACF of an AR(2) process decays to zero slowly, not as fast as it is for the MA process.
- PACF instead, vanishes after the second lag:

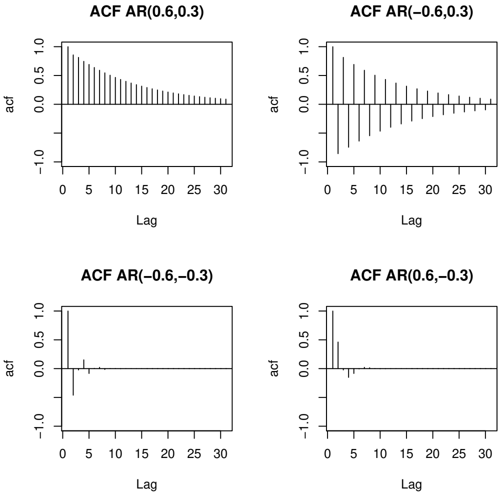
$$\phi_{11} = \rho(1)$$

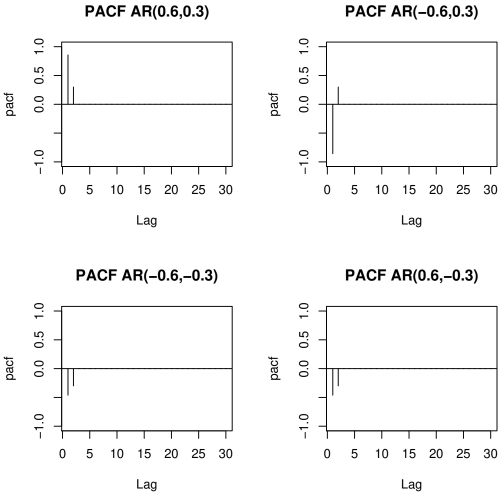
$$\phi_{22} = \varphi_2$$

and

$$\phi_{hh} = 0 \quad \textit{for} \;\; h \geq 3.$$

- Let's have a look at ACF and PACF...


ACF AR(0.6,0.3)
ACF AR(−0.6,0.3)
ACF AR(−0.6,−0.3)
ACF AR(0.6,−0.3)
acf
Lag
−1.0
0.0
0.5
1.0
0
5
10
15
20
25
30


PACF AR(0.6,0.3)
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PACF AR(−0.6,−0.3)
PACF AR(0.6,−0.3)
pacf
Lag
−1.0
0.0
0.5
1.0
0
5
10
15
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25
30

AR(p): Autoregressive of order p

$$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \ldots + \varphi_p X_{t-p} + \epsilon_t$$

$$\epsilon_t \sim WN(0, \sigma^2)$$

$$(X_t = \mu + \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \ldots + \varphi_p X_{t-p} + \epsilon_t).$$

- The model can be written as

$$(1 - \varphi_1 B - \varphi_2 B^2 - \ldots - \varphi_p B^p) X_t = \epsilon_t.$$

- Stationarity requires the roots in in B of the equation

$$(1 - \varphi_1 B - \varphi_2 B^2 - \ldots - \varphi_p B^p) = \Phi(B) = 0$$

to lie outside the unit circle.

- As for the $AR(2)$, we have

$$\mathbb{E}(X_t) = 0,$$

$$\gamma(h) = \varphi_1\gamma(h-1) + \varphi_2\gamma(h-2) + \ldots + \varphi_p\gamma(h-p) \quad \textit{for} \;\; h = 1, 2, \ldots$$

- The ACF is

$$\rho(h) = \varphi_1\rho(h-1) + \varphi_2\rho(h-2) + \ldots + \varphi_p\rho(h-p) \quad \textit{for} \;\; h = 1, 2, \ldots$$

- The ACF of a stationary porcess of order p decays to zero exponentially or in sinusoidal mode depending on whether the roots are real or complex.
- The PACF is such that:

$$\phi_{hh} = 0 \quad \textit{for } h > p.$$