Time Series Analysis

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Class 9

Duality AR and MA processes

1 A stationary $AR(p)$ process admits $MA(\infty)$ representation:

$$(1 - \varphi B)X_t = \epsilon_t \quad \Rightarrow \quad X_t = (1 - \varphi B)^{-1}\epsilon_t.$$

- Viceversa holds as well, that is an invertible $MA(q)$ process has AR(∞) representation.
- Consider the simple case of an invertible $MA(1)$:

$$X_t = (1 + \theta B)\epsilon_t,$$

or,

$$\epsilon_t = (1 + \theta B)^{-1}X_t.$$

- The polynomial

$$(1+\theta B)^{-1} = (1-(-\theta B))^{-1} = \frac{1}{1-(-\theta B)} = \sum_{i=0}^{\infty}(-\theta B)^i,$$

- from which,

$$\epsilon_t = (1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \ldots)X_t,$$

  or,

$$X_t = \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 X_{t-3} + \ldots + \epsilon_t \quad AR(\infty).$$

- Notice that this holds true only if $|\theta| < 1$.

2. Stationary $AR(p)$ processes have ACF that goes to zero fast, and PACF that goes to zero after lag p.

   Invertible $MA(q)$ processes have ACF that goes to zero after lag q and ACF that goes to zero fast.
3. The parameters of an $AR(p)$ are required that the roots of $\Phi(B) = 0$ lie outside the unit circle for the process to be stationary. No rquirements for invertibility.

   The parameters of an $AR(p)$ are required that the roots of $\Theta(B) = 0$ lie outside the unit circle for the process to be invertibility. No requirements for stationarity.

ARMA processes

- In order to reduce the number of parameters in the model it is useful to introduce autoregressive moving averge processes, that is processes composed by both the autoregressive part and the moving average part:

$$\Phi(B)X_t = \Theta(B)\epsilon_t \quad con\ \epsilon_t \sim WN(0, \sigma^2).$$

- In the general form, the $ARMA(p,q)$ writes

$$(1 - \varphi_1 B - \varphi_2 B^2 - \ldots - \varphi_p B^p)X_t = (1 + \theta_1 B + \theta_2 B^2 + \ldots \theta_q B^q)\epsilon_t,$$

or

$$X_t = \varphi_1 X_{t-1} + \ldots + \varphi_p X_{t-p} + \theta_1 \epsilon_{t-1} + \ldots \theta_q \epsilon_{t-q} + \epsilon_t.$$

- A necessary and sufficient condition for an *ARMA* process to be stationary is to have its *AR* part stationary, that is the roots of the equation $\Phi(B) = 0$ lie outside the unit circle.
- A necessary and sufficient condition for an *ARMA* process to be invertible is to have its *AR* part stationary, that is the roots of the equation $\Theta(B) = 0$ lie outside the unit circle.
- If X_t is stationary, then its autoregressive part can be inverted. The process can be written in terms of the MA(∞) representation:

$$X_t = \frac{\Theta(B)}{\Phi(B)} \epsilon_t.$$

- Let's see the simplest form

$$(1 - \varphi B)X_t = (1 + \theta B)\epsilon_t.$$

- We know the if $|\varphi| < 1$, then

$$X_t = \frac{(1 + \theta B)}{(1 - \varphi B)}\epsilon_t.$$

- It can be written with the MA(∞) representation.

$$X_t = \epsilon_t + \sum_{i=1}^{\infty} (\theta + \varphi)\varphi^{i-1}\epsilon_{t-i}.$$

- Notice in the expression the quantity $(\theta + \varphi)$ that impose no choice of parameters such that $(\theta \neq -\varphi)$.
- It is necessary that the roots of $\Theta(B) = 0$ are not equal and of opposite sign with respect to those of $\Phi(B) = 0$ (common factor).

- Considering the $MA(\infty)$ representation of the $ARMA(p, q)$,

$$\mathbb{E}(X_t) = 0$$

- For the ACF consider the simple case of an $ARMA(1, 1)$

$$X_t = \varphi X_{t-1} + \theta\epsilon_{t-1} + \epsilon_t.$$

- Multiplying by X_{t-h} and taking expectation gives

$$\mathbb{E}(X_{t-h}X_t) = \varphi\mathbb{E}(X_{t-h}X_{t-1}) + \theta\mathbb{E}(X_{t-h}\epsilon_{t-1}) + \mathbb{E}(X_{t-h}\epsilon_t).$$

$$\gamma(h) = \varphi\gamma(h-1) + \theta\gamma_{\epsilon_{t-1}x_{t-h}} + \gamma_{\epsilon_t x_{t-h}}.$$

- For $h = 0$

$$\gamma(0) = \varphi\gamma(1) + \theta\varphi\sigma^2 + \theta^2\sigma^2 + \sigma^2 = \varphi\gamma(1) + \sigma^2(1 + \theta(\theta + \varphi)).$$

- For $h = 1$

$$\gamma(1) = \varphi\gamma(0) + \theta\sigma^2.$$

- For $h \geq 2$

$$\gamma(h) = \varphi\gamma(h-1).$$

- that is,

$$\gamma(h) = \varphi^{h-1}\gamma(1).$$

- That is,

$$\gamma(0) = \frac{1 + 2\varphi\theta + \theta^2}{1 - \varphi^2}\sigma^2$$

$$\rho(1) = \frac{(\varphi + \theta)(1 + \varphi\theta)}{1 + 2\varphi\theta + \theta^2}$$

$$\rho(h) = \varphi\rho(h-1) \qquad h \geq 2.$$

- Therefore, an $ARMA(1,1)$ process has a complicated ACF, similar to that of the AR only after $h = 2$.

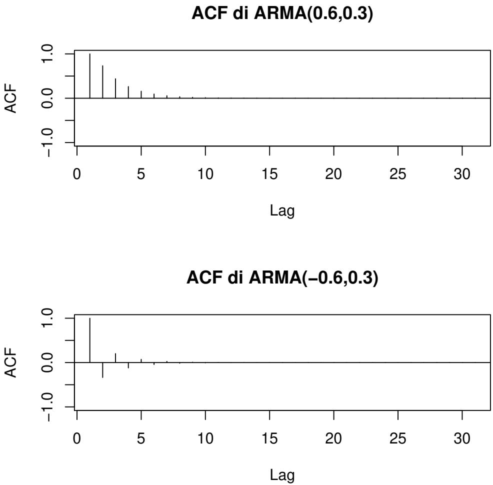

Figure: ACF of stationary and invertible $ARMA(1,1)$.

- PACF only consists of a single value

$$\phi_{11} = \rho(1).$$

- For $k > 1$ the value ϕ_{kk} is computed by means of the general PACF equation.
- PACF of $ARMA(1, 1)$ behaves like that of the $MA(1)$.

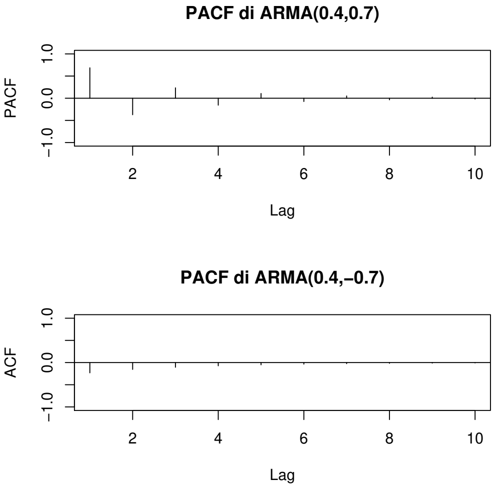

Figure: PACF of stationary and invertible $ARMA(1,1)$.

- The ACF of a stationary and invertible $ARMA(p,q)$ process behaves like that of an $AR(p)$ after lag q.
- The PACF of a stationary and invertible $ARMA(p,q)$ process behaves like that of an $MA(q)$ after lag p.
- No roots of the characteristic polynomial of the AR part should be equal and have opposite sign with respect to those of the MA part.
- Otherwise, the process would be identifiable as an $ARMA(p\text{-}1, q\text{-}1)$.
- Plots are useful to help choosing the values of p and q in mixed models (It is rare to find models with order higher than $ARMA(2,2)$).
- If ACF and PACF go to zero, it is convenient to estimate mixed models by considering a growing order of parameters (we will discuss it in the course).
- Identifiability is not an easy task and requires statistical tools.