Stochastic Processes 2022/23: weekly exercises

The following weekly collection of exercises has been prepared for the students of the course of Stochastic Processes, master degree in Finance and Insurance, Sapienza University of Rome.

Most of the exercises have used as *exam exercises* during the years 2021 and 2022.

Some exercises have been taken from the following books

- G. Grimmet, D. Welsh (2014) Probability, an introduction (second edition). [**GW2014**]
- G. Grimmet, D. Stirzaker (2001) One thousand exercises in probability. [**GS2001**]
- S. Ross. Introduction to probability models.

The material is under construction. New exercises and solutions will be added continuously. This version has been published on **January 16, 2023**. Updated versions will be published until the date of the January exam (January 19, 2023)

Stochastic processes exercises: week 1

- Finite probability
- Enumeration
- Permutations and combinations

1. Three friends take a train with four coaches. They randomly choose the coach. What is the probability that they sit in the same coach?

2. In a game of bridge, the 52 cards of a conventional pack are distributed at random between the four players in such a way that each player receives 13 cards. (**GW2014, Exercises 1.27**)

   (a) Calculate the probability that each player receives one ace.

   (b) Calculate the probability that two given hands in bridge contain k aces between them

   (c) Calculate the probability that a hand in bridge contains 6 spades, 3 hearts, 2 diamonds and 2 clubs

3. You distribute n balls into 3 empty boxes U_1, U_2, U_3 at random and independently. (Each box has the same probability to to receive a ball and multiple occupancy being permitted)

   Find the probability that

   (a) the box U_1 is empty;

   (b) the boxes U_1 and U_2 are empty;

   (c) two boxes are empty

   (d) only the first box is empty

   (e) at least one of the boxes is empty

   (**PS21, january**)

4. You have three red balls (R) and three white balls (B). The balls are randomly thrown into three boxes (for each throw the three boxes have the same probability to receive the ball and the throws are independent)

   (a) What is the probability that each box contains exactly a red ball and a white ball?

   (b) What is the probability that all the three red balls are thrown the same box?

   (c) Now modify the experiment and suppose to have three balls of the same color and five boxes. What is the probability that no box contains more than one ball?

5. There are 6 students and 6 seats where they can seat. They are 3 males and 3 females.

(a) In how many ways they can seat?
(b) In how many ways they can seat so that all the males are in a row and the femals also?
(c) In how many ways they can seat so that the males are in a row?
(d) In how many ways they can seat so that 2 students of the same sex are not back to back?

6. Consider the experiment where a coin is tossed n times
   (a) Find the cardinality of Ω
   (b) What is the number of outcomes of Ω in which we have exactly k heads (Ex 1.25)
   (c) If all possible outcomes are equiprobable (the coin is fair) what is the probability of getting k heads? And that of getting at least k heads?

7. We distribute r indistinguishable balls int n cells at random, multiple occupancy can be permitted,
   (a) How many arrangements there are?
   (b) How many arrangements there are in which the first cell contains exactly k balls
   (c) What is the probability that the first cell contains exactly k balls

8. A box contains 8 balls enumerated from 1 to 8. You extract 4 balls without replacement
   (a) Calculate the probability that the maximum score is 6
   (b) Calculate the probability that 6 is the second higher value?

9. Fifteen students are to be randomly distributed among three groups of five. Twelve students are female and 3 are males.
   (a) How many different groups of students can be formed
   (b) What is the probability that the three men are in the same group
   (c) What is the probability that in each group there is a man?

Some solutions

1 There are $4 \cdot 4 \cdot 4 = 4^3$ possible choices for the three friends. That is $|\Omega| = 4^3$. Moreover, the three friends may sit in the same coach if they sit all in the first coach, all in the second, all in the third or all in the fourth. Hence there are 4 outcomes ω_i such that the three friend sit in the same coach and

$$P(\text{the three friends sit in the same coach}) = \frac{4}{4^3} = \frac{1}{16}$$

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3 Let E_i be the event "The i-th box is empty".

a We have 3^n arrangements for the n balls and 2^n arrangements with U_1 empty. Then

$$P(E_1) = 2^n/3^n$$

b In this case all the balls must go in U_3 and we have just 1 arrangement of the n balls.

$$P(E_1 \cap E_2) = P(U_3 \text{ is full}) = 1/3^n$$

c We have three possibilities to select the full box. Hence

$$P(\text{ two boxes will remain empty}) = 3/3^n$$

d There are 2^n arrangements where U_1 is empty. From these 2^n outcomes we need to exclude the arrangement where U_2 is also empty and the arrangement where U_3 is empty. Hence we have $2^n - 2$ arragements where only U_1 is empty and

$$P(\text{only the first box is empty}) = \frac{2^n - 2}{3^n}$$

e

$$\begin{aligned}
P(\text{at least one is empty}) &= P(E_1 \cup E_2 \cup E_3) \\
&= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) \\
&- P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) \\
&= 3P(E_1) - 3P(E_1 \cap E_2) + 0 \\
&= 3\left(\frac{2}{3}\right)^n - 3\left(\frac{1}{3}\right)^n
\end{aligned}$$

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5 (a) $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6! = 720$

(b) $3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 2 = 3! \cdot 3! \cdot 2 = 72$ (the sequences can be MMMFFF or FFFMMM)

(c) $4 \cdot 3! \cdot 3! = 144$ (MMMFFF, FMMMFF,FFMMMF,FFFMMM)

(d) $3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 2 = 3! \cdot 3! \cdot 2 = 36$ (MFMFMF, FMFMFM)

Stochastic processes exercises: week 2

- Conditional probability
- Independence
- Discrete random variables

1. You are presented with two urns. Urn I contains 3 white and 4 black balls, and Urn II contains 2 white and 6 black balls.

   (a) You pick a ball randomly from Urn I and place it in Urn II. Next you pick a ball randomly from Urn II. What is the probability that the ball is black?

   (b) This time, you pick an urn at random, each of the two urns being picked with probability 1 , and 2 you pick a ball at random from the chosen urn. Given the ball is black, what is the probability you picked Urn I?

2. A single card is removed at random from a deck of 52 cards. From the remainder we draw two cards at random and find that they are both spades. What is the probability that the first card removed was also a spade?

3. N dice are rolled where N is a discrete Uniform random variable that takes values on $\{1, 2, \ldots, 10\}$ Let S_i be the result for the ith die and let $S = \sum_{i=1}^{N} S_i$ be the total sum.

   (a) Calculate $P(N = 2|S = 4)$

   (b) Calculate $P(S = 4|N \text{ is even})$

   (c) Calculate $P(S = 4, S_1 = 1|N = 2)$

4. You throw three regular dice. Let S be the random variable indicating the total score

   (a) Indicate the set of values taken by S.

   (b) Find $P(S = 10)$

   (c) Find $P(S = 10|\text{Dice A is equal to } 4)$

5. A box contains c colored balls each of a different color. Consider a sequence of independent draws with replacement from the box.

   (a) Calculate the probability that the first two balls have the same colour

   (b) Let Y_1 be the r.v indicating the number of extractions necessary to observe a ball of a different color from that observed in the first extraction. (Exclude the first draw from the count). What is the distribution of Y_1

   (c) Let Y_j be the r.v indicating the number of extractions required to observe a new color, after j have been observed. What is the distribution of Y_j?

   (d) Calculate the probability that in n extractions a given color is never observed.

6. Three students, A, B, C, have equal claims to receive an award. Then, they decide that each will toss a coin, and that the man whose coin falls unlike the other two wins. If all three coins fall alike, they toss again.

   (a) Describe a sample space for the result of the first toss of the three coins, and assign probabilities to its elements.

   (b) What is the probability that A wins on the first toss? That B does? That C does? That there is no winner on the first toss?

   (c) Given that there is a winner on the first toss, what is the probability that it is A?

   (d) Find the marginal probability that A eventually wins the award.

   (e) Given that A wins, what is the distribution of the number of tosses which will be necessary to end the game?

7. Three players, A, B, and C, take turns to roll a die; they do this in the order AB-CABCA...

   (a) Describe the sample space of the first turn reporting the probability for each possible outcome

   (b) Calculate the probability that, of the three players, A is the first to throw a 6

   (c) Calculate the probability that, of the three players, A is the first to throw a 6, B the second, and C the third

   (d) Calculate the probability that the first 6 to appear is thrown by A, the second 6 to appear is thrown by B, and the third 6 to appear is thrown by C

8. Assume that each child who is born is equally likely to be a boy or girl. If a family has two children

   (a) what is the probability that both are girls given that the eldest is a girl

   (b) what is the probability that both are girls given that at least one is a girl

9. Two dice are rolled

   (a) What is the probability that at least one is six

   (b) If the two faces are different, what is the probability that at least one is six

10. Bill and George go target shooting together. Both shoot at a target at the same time. Suppose Bill hits the target with probability 0.7, whereas George, independently, hits the target with probability 0.4.

    (a) Given that exactly one shot hit the target, what is the probability that it was George's shot?

    (b) Given that the target is hit, what is the probability that George hit it?

11. Eye color is determined by a single pair of genes. If both genes received from the parents are those of light-eyes (C) then the child will have light eyes. If both genes are dark-eye (S) genes or the child receives one type C and one type S, then the child will have dark eyes. Each newborn receives a gene from each parent independently and each gene is randomly chosen between the two possessed by each parent. Giacomo has dark eyes (S), and both of his parents have dark eyes but Giacomo's sister has light eyes (C).

    A. What is the probability the Giacomo has a gene of type C?

    Suppose that Giacomo is married with a woman with light eyes

    B. What is the probability that the first child has light eyes?
    C. If the first child has dark eyes, what is the probability that the second child has light eyes?

12. The probability that any child in a certain family will have blue eyes is $1/4$, and this feature is inherited independently by different children in the family. Suppose that the family have five children

    A. What is the most probable number of children with blue eyes?
    B. If it is known that at least one of these children has blue eyes, what is the probability that at least three of the children have blue eyes?
    C. If it is known that the youngest child in the family has blue eyes, what is the probability that at least three of the children have blue eyes?
    D. Explain why the answer in part (C) is different from the answer in part (B).

13. A smoker has two match boxes where there are 5 matches. Every time the smoker lights up a cigarette he takes a match randomly from one of the boxes and he throws it away. Let X be the number of matches in the non empty box when the first box is empty. Write the probability mass function of X

14. A fair die is thrown n times. Show that the probability that there are an even number of sixes is $\frac{1}{2}\left(1+\left(\frac{2}{3}\right)^n\right)$. Consider 0 as an even number. ([**GW14, Problem 1.11-1**])

Some solutions

2 Let S_i be the event "the ith card is spade" Note that

$$\begin{aligned} P(S_1 \cap S_2 \cap S_3) &= P(S_1)P(S_2|S_1)P(S_3|S_1, \cap S_2) \\ &= \frac{13}{52}\frac{12}{51}\frac{11}{50} = \frac{\binom{13}{3}}{\binom{52}{3}} \end{aligned}$$

$$P(S_2 \cap S_3) = \frac{\binom{13}{2}}{\binom{52}{2}} = \frac{13}{12}\frac{52}{51}$$

Then

$$P(S_1|S_2, S_3) = \frac{P(S_1 \cap S_2 \cap S_3)}{P(S_2 \cap S_3)} = \frac{\frac{13}{52}\frac{12}{51}\frac{11}{50}}{\frac{13}{12}\frac{52}{51}} = \frac{11}{50}$$

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8 Let G_1 be the the event "the first child is a girl" (this is the event "the eldest is a girl) and G_2 be the event "the second child is a girl". Note that G_1 and G_2 can be assumed independent and that $(G_1 \cap G_2) \cap G_1 = G_1 \cap G_2$

Then for (a) we have that

$$P(G_1 \cap G_2|G_1) = \frac{P((G_1 \cap G_2) \cap G_1)}{P(G_1)} = \frac{P(G_1 \cap G_2)}{P(G_1)} = \frac{(1/2)(1/2)}{1/2} = \frac{1}{2}$$

or alternatively note that

$$P(G_1 \cap G_2|G_1) = P(G_2|G_1) = \frac{1}{2} \quad \textit{by independence}$$

For point (b) note that at least one girl is the event $G_1 \cup G_2$ and $G_1 \cap G_2 \subset G_1 \cup G_2$. Remember that if $A \subseteq B, P(A \cap B) = P(A)$. Then

$$P(G_1 \cap G_2|G_1 \cup G_2) = \frac{P((G_1 \cap G_2) \cap (G_1 \cup G_2))}{P(G_1 \cup G_2)} = \frac{P(G_1 \cap G_2)}{P(G_1 \cup G_2)} = \frac{1/4}{3/4} = \frac{1}{3}$$

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9 Let D_i be the result of the $i - th$ die. For point (a) note that

$$\begin{aligned} P(\text{at least one six}) &= P(D_1 = 6 \cup D_2 = 6) \\ &= P(D_1 = 6) + P(D_2 = 6) - P(D_1 = 6 \cap D_2 = 6) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36} \end{aligned}$$

For point (b) note that the intersection between the event the two faces are different and at least one six is given by the 10 outcomes

$$\{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)\}$$

Moreover, the number of outcomes with different faces is 30, that is 36-6. In fact there are 6 outcomes with the same face: $(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)$ Then

$$P(\text{at least one six}|\text{the two faces are different}) = 10/30 = 1/3$$

More formally,

$$\begin{aligned}
P(\text{at least one six}|\text{the two faces are different}) &= P(D_1 = 6 \cup D_2 = 6 | D_1 \neq D_2) \\
&= \frac{P((D_1 = 6 \cup D_2 = 6) \cap (D_1 \neq D_2))}{P(D_1 \neq D_2)} \\
&= \frac{P\left(D_1 = 6 \cap (D_1 \neq D_2) \bigcup D_2 = 6 \cap (D_1 \neq D_2)\right)}{30/36} \\
&= \frac{P(D_1 = 6 \cap D_2 \neq 6 \bigcup D_1 \neq 6 \cap D_2 = 6)}{30/36} \\
&= \frac{P(D_1 = 6 \cap D_2 \neq 6) + P(D_1 \neq 6 \cap D_2 = 6)}{30/36} \\
&= \frac{\frac{1}{6} \cdot \frac{5}{6} + \frac{5}{6}\frac{1}{6}}{30/36} = \frac{10/36}{30/36} = \frac{1}{3}
\end{aligned}$$

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10 Let B be the event "Bob hits the target" and G be the event "George hits the target". The event "exactly one shot hit the target is " $B \cap G^c \bigcup B^c \cap G$. By independence we have that $P(B \cap G) = 0.7 \cdot 0.4$, $P(B \cap G^c) = 0.7 \cdot 0.6$ and $P(B^c \cap G) = 0.3 \cdot 0.4$. Moreover $B \cap G^c$ and $B^c \cap G$ are disjoint then

$$P\left(B \cap G^c \bigcup B^c \cap G\right) = 0.7 \cdot 0.6 + 0.3 \cdot 0.4$$

and

$$\begin{aligned}
P\left(G | B \cap G^c \bigcup B^c \cap G\right) &= \frac{P(G \cap (B \cap G^c \cup B^c \cap G))}{P(B \cap G^c \bigcup B^c \cap G)} \\
&= \frac{P(B^c \cap G)}{0.7 \cdot 0.6 + 0.3 \cdot 0.4} = \frac{0.3 \cdot 0.4}{0.7 \cdot 0.6 + 0.3 \cdot 0.4}
\end{aligned}$$

For point (b)

$$P(G|G \cup B) = \frac{P(G \cap (G \cup B))}{P(G \cup B)} = \frac{P(G)}{P(G \cup B)} = \frac{0.4}{0.7 + 0.4 - 0.7 \cdot 0.4}$$

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13 Note $X \in \{1,2,3,4,5\}$. To have $X = k$ we have smoked 4 cigarettes to have a box with only a match and 5-k cigarettes to have the other with exactly k matches. Finally we need to take a match from the box with only one match. Then

$$\begin{aligned} P(X = k) &= 2\binom{4+5-k}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{4+5-k-4}\frac{1}{2} \\ &= \binom{9-k}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-k} \end{aligned}$$

Observe that

$$\begin{aligned} P(X=1) &= \binom{8}{4}\frac{1}{2^8} = 0.2734 \\ P(X=2) &= \binom{7}{4}\frac{1}{2^7} = 0.2734 \\ P(X=3) &= \binom{6}{4}\frac{1}{2^6} = 0.2344 \\ P(X=4) &= \binom{5}{4}\frac{1}{2^5} = 0.1562 \\ P(X=5) &= \binom{4}{4}\frac{1}{2^4} = 0.0625 \end{aligned}$$

and the probabilities sum to 1

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14 Let A be the event "an even number of sixes" and let Y counting the number of sixes be a Binomial random variable with $p = 1/6$ and size n. Remeber to consider 0 as an even number.

$$\begin{aligned} P(A) &= \sum_{y=0, y\, even}^{n} P(Y=y) = \sum_{y=0, y\, even}^{n}\binom{n}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{n-y} \\ &= \frac{1}{2}\left(\sum_{y=0,}^{n}\binom{n}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{n-y} + \sum_{y=0}^{n}(-1)^y\binom{n}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{n-y}\right) = * \end{aligned}$$

In fact, in the last formula the probabilities for the even numbers are summed two times while those for the odd numbers cancel out. Then

$$\begin{aligned} * &= \frac{1}{2}\left(1 + \sum_{y=0,}^{n}\binom{n}{y}\left(-\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{n-y}\right) \\ &= \frac{1}{2}\left(1 + \left(-\frac{1}{6}+\frac{5}{6}\right)^n\right) = \frac{1}{2}\left(1 + \left(\frac{2}{3}\right)^n\right) \end{aligned}$$

Stochastic processes exercises: week 3

- Discrete random variables: mean and variance
- Multivariate discrete random variables
- Transformations of discrete random variables

1. An urn contains five red, three orange, and two blue balls. Two balls are randomly selected. Let X represent the number of orange balls
   (a) Find the p.m.f. of X
   (b) Calculate $E(X)$
   (c) Calculate $Var(X)$
2. Suppose there are three students and their hats are returned randomly with each of the 3! permutations equally likely. Let X be the number of hats returned to the correct owner
   (a) Find the probability mass function of X, the mean $E(X)$ and the variance $Var(X)$
   (b) Find the probability mass function of $Y = X^2$ and the mean $E(Y)$
3. Suppose there are four students, two girls and two boys, and their hats are returned randomly with each of the 4! permutations equally likely. Let X be the total number of hats correctly returned to the male owners and Y the total number of hats correctly returned to the female owners and $Z = X + Y$ the total number of hats correctly returned
   (a) Find the probability mass function of Z and that of (X, Y)
   (b) Find $E(X)$, $E(Y)$ $E(Z)$
   (c) Are X and Y independent?
   (d) Find the conditional distribution of $Z|X = 1$
4. Suppose there are four students, two girls and two boys, and their hats are returned randomly with each of the 4! permutations equally likely. Let X be the total number of hats belonging to one of the boys returned to a boy (even he is not the owner) and Y the total number of hats belonging to a girl returned to a girl (even if she is not the owner) and $Z = X + Y$ the total number of hats correctly returned
   (a) Find the probability mass function of (X, Y)
   (b) Find $E(X)$, $E(Y)$ $E(Z)$
   (c) Are X and Y independent?
   (d) Find the probability mass function of (X, Z). Are X and Z dependent?

5. The discrete joint distribution of the lifetimes X and Y of two connected components in a machine can be modelled by

$$P(X = k, Y = j) = \frac{1}{e^2} \frac{1}{k!(j-k)!}$$

for $k = 0, 1, 2, \ldots$ and $j = k, k+1, k+2, \ldots$

   (a) Find the marginal distribution of X and Y

   (b) Find the joint distribution of X and $Z = Y - X$. Are X and Z independent?

   (c) Find the correlation between X and Y

6. Let (N, Y) be such that $Y|N = n \sim Binomial(n, p)$ and $N \sim \text{Poisson}(\lambda)$.

   (a) Write the probability mass function of the double random variable. Specify the points where the probability mass function is positive

   (b) Find the marginal distribution of Y

   (c) Find the conditional distribution of $N|Y = y$

7. N dice are rolled where N is a random variable with $P(N = 1) = P(N = 2) = 0.5$. Let S_i be the result for the ith die and let $S = \sum_{i=1}^{N} S_i$ be the total sum.

   (a) Write the probability mass function of the random variable (N, S)

   (b) Find the probability mass function of te random variable S. Calculate $E(S)$

   (c) Now suppose that N is a discrete Uniform random variable that takes values on $\{1, \ldots, 10\}$. Find $E(S)$

8. There are k people ina room. Assume each birthday is equally probable to be any of the 365 days of the year and that peple's birthday are independent.

   (a) What is the probability that at least one pair of pepole in the group have the same birthday?

   (b) What is the probabibility that a given pair has the same birthday

   (c) If $k = 50$ what is the expected number of pairs with the same birthdays

9. An urn contains a red ball and a blue ball. We take a ball at random. If it is blue we stop. If it is red we put it in the urn with another red ball. Suppose to stop this sequence after 10 iterations or when we obtain the first blue ball. Let X be numeber of iterations.

   (a) Find the p.m.f. of X

   (b) Find $E(X)$

   (c) Now suppose that we do not stop the extractions after 10 iterations.Show that $P(X = \infty) = 0$ but $E(X) = \infty$

Some solutions

1 $X \in \{0, 1, 2\}$

$$\begin{aligned} P(X=0) &= \frac{\binom{7}{2}}{\binom{10}{2}} = \frac{7\cdot 6}{10\cdot 9} \\ P(X=1) &= \frac{\binom{3}{1}\binom{7}{1}}{\binom{10}{2}} = 2\cdot\frac{3}{10}\cdot\frac{7}{9} \\ P(X=2) &= \frac{\binom{3}{2}}{\binom{10}{2}} = \frac{3}{10}\cdot\frac{2}{9} \end{aligned}$$

$$E(X) = 0\cdot\frac{7\cdot 6}{10\cdot 9} + 1\cdot 2\cdot\frac{3}{10}\cdot\frac{7}{9} + 2\cdot\frac{3}{10}\cdot\frac{2}{9} = \frac{54}{90} = 9/15 = 3/5 = 0.6$$

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3 There are $4! = 24$ possible orderings for the returning hats. Note that $Z = 0$ is given by the 9 sequences

$(2,1,4,3), (2,3,1,4), (2,4,2,3), (3,1,4,2), (3,4,1,2), (3,4,2,1), (4,1,2,3), (4,3,1,2), (4,3,2,1)$

$Z = 1$ is given by the 8 sequences

$$(1,3,4,2), (1,4,2,3), (3,2,4,1), (4,2,1,3), (2,4,3,1), (4,1,3,2), (2,3,1,4)(3,1,2,4)$$

$Z = 2$ is given by the the 6 sequences

$$(1,2,4,3), (1,4,3,2), (1,3,2,4), (4,2,3,1), (3,2,1,4), (2,1,3,4)$$

$Z = 3$ cannot be obtained while $Z = 4$ is given only by

$$(1,2,3,4)$$

The p.m.f. of Z is

z	$p_Z(z)$
0	9/24
1	8/24
2	6/24
4	1/24

By counting the true matches in the first two positions and in the second two positions of all the 24 possible orderings we have that the p.m.f. of (X, Y) is

X \ Y	0	1	2
0	9/24	4/24	1/24
1	4/24	4/24	0
2	1/24	0	1/24

$$E(X) = E(Y) = 0 \cdot \frac{14}{24} + 1 \cdot \frac{8}{24} + 2 \cdot \frac{2}{24} = \frac{1}{2}$$

$$E(Z) = 0 \cdot \frac{9}{24} + 1 \cdot \frac{8}{24} + 2 \cdot \frac{6}{24} + 4\frac{1}{24} = \frac{24}{24} = 1$$

Note that $P(X = 1, Y = 1) = 0 \neq P(X = 1)P(Y = 1)$. Hence X and Y are dependent.

Note that

$$P(Z = 0|X = 1) = 0$$

$$P(Z = 1|X = 1) = \frac{P(X = 1, Y = 0)}{P(X = 1)} = \frac{4/24}{8/24} = \frac{1}{2}$$

$$P(Z = 2|X = 1) = \frac{P(X = 1, Y = 1)}{P(X = 1)} = \frac{4/24}{8/24} = \frac{1}{2}$$

The p.m.f. of $Z|X = 1$ is

z	$p_{Z\|X=1}(z)$
1	1/2
2	1/2

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9 By assuming that the sequence terminates after 10 extractions we have $X \in \{0, 1, \ldots, 10\}$ Let us indicate with R_i the event "the ith ball extracted is red" and with B_i the event "the ith ball extracted is blue". Note that

$$\begin{aligned}
P(X = 1) &= P(B_1) = \frac{1}{2} \\
P(X = 2) &= P(R_1 \cap B_2) = P(R_1)P(B_2|R_1) = \frac{1}{2}\frac{1}{3} \\
P(X = 3) &= P(R_1 \cap R_2 \cap B_3) = P(R_1)P(R_2|R_1)P(B_3|R_1, R_2) = \frac{1}{2}\frac{2}{3}\frac{1}{4} = \frac{1}{3}\frac{1}{4} \\
P(X = 4) &= P(R_1 \cap R_2 \cap R_3 \cap B_4) = P(R_1)P(R_2|R_1)P(R_3|R_1, R_2)P(B_4|R_1, R_2, R_3) = \frac{1}{2}\frac{2}{3}\frac{3}{4}\frac{1}{5} \\
&= \frac{1}{4}\frac{1}{5}
\end{aligned}$$

In general, for $k = 1, 2, \ldots, 9$

$$P(X = k) = \prod_{j=1}^{k-1} \frac{j}{j+1} \cdot \frac{1}{k+1} = \frac{(k-1)!}{k!} \cdot \frac{1}{k+1} = \frac{1}{k} \cdot \frac{1}{k+1} = \frac{1}{k} - \frac{1}{k+1}$$

Note also that after 9 red balls we terminate the sequence both with the evenr R_{10} and with the event B_{10}, then

$$P(X = 10) = P(R_1 \cap R_2 \cdots \cap R_9) = \frac{1}{2}\frac{2}{3}\frac{3}{4}\frac{4}{5}\frac{5}{6}\frac{6}{7}\frac{7}{8}\frac{8}{9}\frac{9}{10} = \frac{1}{10}$$

In fact

$$\begin{aligned}
P(X=10) &= 1-\sum_{k=1}^{9}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1-\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{8}-\frac{1}{9}+\frac{1}{9}-\frac{1}{10}\right)\\
&= 1-\left(1-\frac{1}{10}\right) = \frac{1}{10}
\end{aligned}$$

The mean of X is then

$$\begin{aligned}
E(X) &= \sum_{k=1}^{10} kP(X=k) = \sum_{k=1}^{9} k\frac{1}{k}\frac{1}{k+1} + 10\frac{1}{10} = \sum_{i=1}^{9}\frac{1}{k+1}+1\\
&= \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{10}+1 = 2.929
\end{aligned}$$

If we do not stop the sequence after 10 iterations, note that

$$P(X\geq n) = 1-\sum_{k=1}^{n-1}P(X=k) = 1-\sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1-\left(1-\frac{1}{n}\right) = \frac{1}{n}$$

Then

$$P(X=\infty) = \lim_{n\to\infty}P(X\geq n) = \lim_{n\to\infty}\frac{1}{n} = 0$$

but

$$E(X) = \sum_{k=1}^{\infty} kP(X=k) = \sum_{k=1}^{\infty} k\frac{1}{k}\frac{1}{k+1} = \sum_{k=1}^{\infty}\frac{1}{k+1} = \sum_{k=2}^{\infty}\frac{1}{k} = \infty \quad \text{(Harmonic series)}$$

Stochastic processes exercises: week 4

- Discrete random variables: mean and variance
- Multivariate discrete random variables
- Transformations of discrete random variables

1. Consider the experiment of tossing a coin three times so that the sample space. Let X be the number of heads in the first two tosses and Y the total number heads on all the three tosses

   (a) Find the probability mass function of X, Y

   (b) Find the mean of $Z = X + Y$

   (c) Find the probability mass function of $Z = X + Y$

2. Suppose that the joint probability mass function of (X, Y) is

$$P(X = i, Y = j) = \binom{j}{i} e^{-2\lambda}\lambda^j / j! \quad 0 \leq i \leq j$$

   (a) Find the probability mass function of Y

   (b) Find the probability mass function of X

   (c) Find the probability mass function of $Y - X$

3. Let $X_1, X_2, \ldots, X_n$ be independent random variables each having mass function

$$P(X_i = k) = \frac{1}{N} \quad k = 1, \ldots N$$

   Find the probability mass function of

$$U_n = \min\{X_1, \ldots, X_n\} \quad V_n = \min\{X_1, \ldots, X_n\}$$

4. Each time you flip a certain coin, heads appears with probability p. Suppose that you flip the coin a random number N of times, where N has the Poisson distribution with parameter λ and is independent of the outcomes of the flips. Find the distributions of the numbers X and Y of resulting heads and tails, respectively, and show that X and Y are independent.

5. Suppose that an urn initially contains r red balls and b blue balls. At each stage a ball is randomly selected from the urn and then is returned with m others of the same color. Let X_k be the number of red balls extracted in the first k selections

   (a) Find $E(X_1)$

   (b) Find $E(X_2)$

   (c) Find $E(X_3)$

   (d) Conjecture the value of $E(X_k)$

Some solutions

4 For this exercise note

$$P(X = x, Y = y) = P(N = x + y, X = x) = P(N = x + y)P(X = x|N = x + y)$$

---

5 $X_1 \sim \{0, 1; b/(b + r), r/(b + r)\}$, then

$$E(X_1) = \frac{r}{b + r}$$

Let R_2 be the r.v. indicating if at the second extraction we have a red ball.

$$X_2 = X_1 + R_2$$

Note that

$$E(R_2) = E(E(R_2|X_1)) = E(r + m \cdot X_1)/(r + b + m)$$

and

$$E(r + mX_1) = \frac{r}{r + b}(r + b + m)$$

hence

$$E(R_2) = \frac{r}{r + b} \qquad E(X_2) = 2\frac{r}{r + b}$$

Let R_3 be the r.v. indicating if at the third extraction we have a red ball.

$$X_3 = X_2 + R_3$$

Note that

$$E(R_3) = E(E(R_2|X_2)) = E(r + m \cdot X_2)/(r + b + 2m)$$

and

$$E(r + mX_2) = \frac{r}{r + b}(r + b + 2m)$$

hence

$$E(R_3) = \frac{r}{r + b} \quad E(X_3) = 3\frac{r}{r + b}$$

$$E(X_k) = k\frac{r}{r + b}$$

Stochastic processes exercises: week 6

- Continuous random variables

1. Let X be a random variable with distribution function

$$F_X(t) = \begin{cases} 0 & t \leq 0 \\ \frac{2}{5}t & 0 < t \leq 1 \\ \frac{3}{5}t - \frac{1}{5} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}$$

   a. Verify that $F_X(\cdot)$ is a distribution function
   b. Verify that X is a continuous random variable and find the density
   c. Without doing any calculations find $P(X \leq 1.5)$.
   d. Consider $Y = X^2/4$. Find the distribution of Y

2. Let X be an exponential random variable with mean θ. Let $Y = \log X$.

   a Find the support of the r.v. Y
   b Find the density of the r.v. Y
   c Find $P(Y > 0)$
   d Find the median of the r.v. Y

3. Let X be a random variable with density

$$f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & otherwise \end{cases}$$

   (a) What is the value of c
   (b) What is the distribution function of X
   (c) What is the expectation of X

4. Let X be a random variable with density

$$f(x) = \begin{cases} c(4x - 2x^2) & 0 < x < 2 \\ 0 & otherwise \end{cases}$$

   (a) What is the value of c
   (b) Find $P(1/2 < X < 3/2)$
   (c) What is the expectation of X

5. Find the distribution function for the random variable X with density

$$f(x) = e^{-x-e^{-x}} \quad -\infty < x < \infty$$

6. An expensive item is being insured against early failure. The lifetime of the item is normally distributed with an expected value of seven years and a standard deviation of two years. The insurance will pay A dollars if the item fails during the first or second year and $A/2$ dollars if the item fails during the third or fourth year. If a failure occurs after the fourth year, then the insurance pays nothing. Let X be the random payment for the insurance

   (a) Write the CDF of X

   (b) Calculate the expected value of X

   (c) How to choose A such that the expected value of the payment per insurance is \$50?

Some solutions

5 Note that

$$\frac{d}{du}e^{-e^{-u}} = e^{-u}e^{-e^{-u}} = e^{-u-e^{-u}}$$

Then

$$F(x) = \int_{-\infty}^{x} e^{-u-e^{-u}} du = e^{-e^{-u}}\Big|_{-\infty}^{x} = e^{-e^{-x}} - \lim_{c\to-\infty} e^{-e^{-c}} = e^{-e^{-x}} - 0 = e^{-e^{-x}}$$

Stochastic processes exercises: week 7

- Bivariate continuous random variables
- Margnal densities
- Independence

1. Let (X, Y) be a random variable with density

$$f(x, y) = \begin{cases} k(x+y) & 0 \le x \le 3, \quad 0 \le y \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

   A. Find the value of k.

   B. Calculate $P(X > 2Y)$

   C. Say if X and Y are independent

2. Suppose that (X, Y) have joint density function

$$f(x, y) = \begin{cases} e^{-x-y} & x > 0\, y > 0 \\ 0 & otherwise \end{cases}$$

   Find $P(X + Y \le 1)$ and $P(X > Y)$

3. Consider the random vector (X, Y) with density

$$f(x, y) = \begin{cases} 2\exp(x+y) & x \le y \le 0 \\ 0 & \text{otherwise} \end{cases}$$

   A. Find the marginal density of X and Y

4. Let (X, Y) be a bivariate continuous random variable with density

$$f_{XY}(x, y) = \begin{cases} c & x^2 - 1 \le y \le 1 - x^2 \\ 0 & \text{otherwise} \end{cases}$$

   a Find the value of c

   b Find the marginal densities $f_X(x)$, $f_Y(y)$.

   c Are X and Y independent?

   d Find $P(X^2 + Y^2 \ge 1)$

5. Let X, Y be a random variable uniformly distributed on the triangle with vertices $(0, 0), (1, 1), (0, 1)$

   a Write the joint density of X, Y. Find the marginal densities of X and Y

6. In the $[0,1]$ interval you choose independently n points uniformly distributed. Let $X_1, X_2, \ldots, X_n$ be the random variables indicating these points. Find the densities of the the following random variable

   a. $U = \min(X_1, X_2, \ldots, X_n)$
   b. $V = \max(X_1, X_2, \ldots, X_n)$
   c. Find the density of the random variable (U, V)

7. Let $X_1, X_2, \ldots, X_n$ be n indipendent exponential random variables with mean λ Find the densities of the the following random variables

   a. $U = \min(X_1, X_2, \ldots, X_n)$
   b. $V = \max(X_1, X_2, \ldots, X_n)$
   c. Find the density of the random variable (U, V)

Stochastic processes exercises: week 8

- Bivariate continuous random variables
- Conditional densities
- Sum of random variables and other transformations
- Expectations

1. Let X, Y be a random variable uniformly distributed on the triangle with vertices $(0,0),(1,1),(0,1)$

   a. Find the joint density of U, Z where $U = X+Y$ and $Z = X-Y$ Are U and Z independent?

2. Let X, Y be a jointly continuous random variabe with density

$$f_{XY}(x,y) = \begin{cases} \frac{1}{4}e^{-\frac{1}{2}(x+y)} & x > 0\, y > 0 \\ 0 & otherwise \end{cases}$$

   (a) Find the joint density of U, V where

$$U = \frac{1}{2}(X-Y) \quad V = Y$$

   (b) Find the marginal density of U

3. The joint density of the random variable (X, Y) is

$$f_{XY}(x,y) = \begin{cases} c\, e^{-x}e^{-y} & 0 < x < y < \infty \\ 0 & otherwise \end{cases}$$

   (a) Find the value of c
   (b) Find the conditional densities $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$
   (c) Find the covariance between X and Y
   (d) Consider the transformation $U = 2X$ and $V = X+Y$. Find the joint density of (U, V)

4. Let X and Y be independent exponential random variables with mean $1/\lambda$.

   a Find the joint density of (U, V) where

$$U = X \quad V = X+Y$$

   b Find the marginal density of V

5. Let X and Y be independent **standard** normal random variables

A. Find the density of $Z = X^2 + Y^2$

B. Find the joint density of $V = X + Y$ and $W = X - Y$

6. Let (X, Y) be a bivariate continuous random variable with density

$$f(x,y) = \begin{cases} k(x+y) & 0 \le x \le 2; 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

a. Find the density of the random variable $Z = XY$

7. Let Z be a random variable with Uniform distribution on $(0, 2\pi)$, i.e. $Z \sim U(0, 2\pi)$. Define

$$R_1 = \cos(Z) \text{ and } R_2 = \sin(Z).$$

a. Prove that $E(R_1 R_2) = E(R_1)E(R_2)$

b. Prove that $\mathrm{Var}(R_1 + R_2) = \mathrm{Var}(R1) + \mathrm{Var}(R_2)$

c. Show that R_1 and R_2 are dependent, notwithstanding the previous results

8. Let (X, Y) be a bivariate continuous random variable with density

$$X|Y = y \sim \text{Exp }(y); \qquad Y \sim \text{Exp}(\nu) \quad \nu > 0.$$

a. Find the marginal density of X

b. Find the expectation of X, if it exists.

c. Find the median of X (i.e. the value q such that $P(X \le q^*) = 0.5$

Stochastic processes exercises: week 9

- Recap about joint random variables and random vectors

1. Let X and Y be independent random variables with densities

$$f_X(x) = \begin{cases} 4e^{-4x} & x > 0 \\ 0 & otherwise \end{cases} \quad f_Y(y) = \begin{cases} e^{y} & y < 0 \\ 0 & otherwise \end{cases}$$

   a Find the density of W, Z where $W = 4X - Y$ and $Z = 4X + Y$ (specify the region where thhis density is positive)

   b Find the the covariance $Cov(W, Z)$

2. Let X and Y be two random variables such that the marginal density of X is

$$f_X(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   and the conditional density $Y|X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} 3y^2/x^3 & 0 \leq y < x \\ 0 & \text{otherwise} \end{cases};$$

   Find

   a. The image of the r.v. Y.

   b. The marginal density of Y.

   c. The conditional density of $X|Y$.

   d. The covariance between X and Y.

3. Let X and Y be independent Uniform$(0, 1)$ random variables. Moreover consider

$$Z = \min(X, Y) \quad W = \max(X, Y)$$

   .

   (a) Find the density of $S = X - Y$

   (b) Find $P(W > 3/4 | Z < 1/4)$.

   (c) Find the distribution of W conditional on the event $Z \leq z$.
   Hint: Write separately the joint distribution $P(W \leq w, Z \leq z)$ for the cases $w \leq z$ and $w > z$.

4. Let X_1 and X_2 be independent, idenntically distributed random variables with density

$$f(x) = \begin{cases} 2x/a^2 & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

where a is a positive constant. Define new random variables Y and Z as

$$Y = \max(X_1, X_2); \qquad Z = \min(X_1, X_2).$$

a. Compute mean and variance of X_1

b. Compute the distribution and the mean of Y and Z.

c. Prove that the joint density of (Y, Z) is

$$g_{YZ}(y, z) = \begin{cases} 8yx/a^4 & 0 < z < y < a \\ 0 & \text{othrwise} \end{cases}$$

Hint: Start by computing $P(Y \leq y \cap Z > z)$

d. Find the mean of $U = Y - Z$.

5. The buses of an urban line pass to a specific stop every 15 minutes starting at 7 (therefore at 7, 7:15, 7:30 and so on). Sara arrives at the bus stop in an instant that distributes itself as a uniform random variable in the interval $[7:00, 7:30]$.

a) Calculate the probability that Sara will wait less then 5 minutes

b) Calculate the probability that Sara will wait more that 10 minutes

c) The mean waiting time at the bus stop

d) Now suppose that bus departures are not deterministic but random, with waiting times in minutes between one bus and another distributed as independent and identically distributed exponential random variables with $\lambda = 1/15$. Calculate the probability that Sara will wait less then 5 minutes

6. Let $X_1, X_2, \ldots, X_n$ be n independent random variables such that for $j = 1, \ldots, n$,

$$EX_j = \mu; \qquad \text{Var}(X) = \sigma^2.$$

Let $\bar{X} = \sum_{j=1}^{n} X_j/n$ be the sample mean

Find

a) Mean and variance of $\bar{X}$

b) The correlation coefficient between $\bar{X}$ and X_1.

7. Let X, Y be a continuous random variable with density

$$f_{XY}(x, y) = 12xy(1 - y) \quad 0 < x < 1, 0 < y < 1$$

a) Calculate the covariance of X, Y.

b) Consider the transformation $S = X$ and $T = XY^2$. Find the set of values where the density of S, T is positive.

c) Find the joint density of S, T.

d) Find the density of T.

8. Let X and Y be independent and Uniform on (0,1). Consider

$$U = |X - Y| \quad V = \min\{X, Y\}$$

(a) Find the marginal density of U and V

(b) Find the support of U, V

(c) Find the joint density of V, W where $W = U + V$

(d) Find the joint density of U, V

Some solutions

8 The covariance is 0. In fact X and Y are independent since the the joint density $f_{XY}(x,y)$ is the product between two functions $g(x)$ and $h(y)$ $\forall x, y \in \mathcal{R}^2$. A possible choice for the function h and g is

$$g(x) = \begin{cases} 12\,x & x \in (0,1) \\ 0 & otherwise \end{cases} \qquad h(y) = \begin{cases} y(1-y) & \in (0,1) \\ 0 & otherwise \end{cases}$$

The set is

$$A = \{s, t : 0 < s < 1, 0 < t < s\}$$

In fact the inverse transformations are

$$X = S \quad Y = \sqrt{\frac{T}{S}}$$

Since $X \in (0,1)$ and $Y \in (0,1)$ the density of S, T is positive on the set

$$A = \{s, t : 0 < s < 1, 0 < \sqrt{\frac{t}{s}} < 1\} = \{s, t : 0 < s < 1, 0 < \frac{t}{s} < 1\} = \{s, t : 0 < s < 1, 0 < t < s\}$$

The Jacobian matrix is

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{2}\sqrt{\frac{t}{s}}\frac{1}{s} & \frac{1}{2\sqrt{st}} \end{bmatrix}$$

and the determinant is $\frac{1}{2\sqrt{st}}$. The joint density of the random variable S, T on the set A is

$$\begin{aligned} f_{ST}(s,t) &= 12\,s\sqrt{\frac{t}{s}}\left(1 - \sqrt{\frac{t}{s}}\right)\frac{1}{2\sqrt{st}} \\ &= 6\left(1 - \sqrt{\frac{t}{s}}\right) \end{aligned}$$

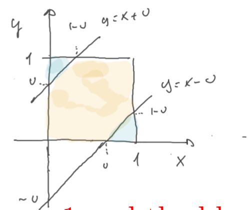
The density of T is

$$\begin{aligned} f_T(t) &= \int_t^1 6\left(1 - \sqrt{\frac{t}{s}}\right) ds = 6\int_t^1 1 ds + 6\sqrt{t}\int_t^1 s^{-1/2} ds \\ &= 6(1-t) - 6\sqrt{t}2(1-\sqrt{t}) = 6t - 12\sqrt{t} + 6 \quad t \in (0,1) \end{aligned}$$

---

9 Note that $0 < U < 1$ and for $u \in (0,1)$

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(-u < X - Y < u) \\ &= P(u > Y - X > -u) = P(u + X > Y > X - u) \\ &= P(X - u < Y < X + u) = P((X,Y) \in \text{yellow area in the figure below}) \\ &= 1 - P((X,Y) \in \text{blue area}) = 1 - (1-u)^2 \end{aligned}$$



In fact, on the square the density is $f_{XY} = 1$ and the blue area is two times $(1-u)(1-u)/2$. Hence

$$f_U(u) = \begin{cases} 2(1-u) & u \in (0,1) \\ 0 & otherwise \end{cases}$$

Note that also $0 < V < 1$ and for $v \in (0,1)$

$$\begin{aligned} F_V(u) &= P(V \leq v) = 1 - P(V > v) = 1 - P(X > v, Y > v) \\ &= 1 - P(X > v)P(Y > v) = 1 - (1-v)^2 \end{aligned}$$

Hence

$$f_V(v) = \begin{cases} 2(1-v) & v \in (0,1) \\ 0 & otherwise \end{cases}$$

To find the support of (U, V) note that $U = \max\{X, Y\} - \min\{X, Y\}$. In fact

$$U = |X - Y| = \begin{cases} X - Y & if\, X > Y \\ Y - X & if\, Y > X \end{cases} = \begin{cases} \max\{X, Y\} - \min\{X, Y\} & if\, X > Y \\ \max\{X, Y\} - \min\{X, Y\} & if\, Y > X \end{cases}$$

Hence $U = \max\{X, Y\} - V$ and since $\max\{X, Y\} > \min\{X, Y\} = V$, when $V = v$ $U \in (0, 1-v)$

To find the distribution of V, W note that $W = U + V = \max\{X, Y\}$. Then $V < W$ and for $v < w$ note that

$$\begin{aligned} F_{VW}(v, w) &= P(V \leq v, W \leq w) = P(W \leq w) - P(V > v, W \leq w) \\ &= P(W \leq w) - P(v < V, W \leq w) \\ &= P(W \leq w) - P(v < X \leq w)^2 = w^2 - (w-v)^2 \end{aligned}$$

The joint density is then

$$f_{VW}(v, w) = \begin{cases} 2 & 0 < v < w < 1 \\ 0 & otherwise \end{cases}$$

That is the joint distribution of minimum and maximum is uniform on the triangle (0,0) (1,1) (0,1) on the plane v, w

Note that we can consider U, V as a transoformation of V, W where $U = W - V$. The absolute value of the Jacobian, $|J| = 1$.

The joint density of U, V is positive on the set $\{u, v : 0 < u < 1 - v, 0 < v < 1\}$, that is

$$f_{UV}(v, w) = \begin{cases} 2 & 0 < u < 1 - v, 0 < v < 1 \\ 0 & otherwise \end{cases}$$

Hence, U, V is uniform on the triangle (0,0) (1,0) (0,1) on the plane u, v

Stochastic processes exercises: week 12

- Markov chains

1. Let $\mathbf{X}$ be a Markov Chain with initial distribution $\lambda = (1, 0, 0)$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

   Find the distribution of the chain at time 2

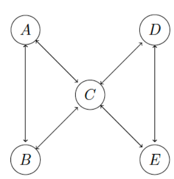
2. Three children, A, B, and C throw a ball between them. When A has the ball, he throws it to B with probability 0.2 or to C with complementary probability. When B has the ball, he throws it to A with probability 0.6 or to C with complementary probability. Finally, when C has the ball, he throws it to A or B with the same probability. Let X_n be the children throwing the ball at time $n = 0, 1, \ldots$

   a. Consider the process as a Markov chain and write the transition matrix of the process.

   b. Suppose that the children throwing the ball at time 0 is uniformly selected, who has the higher probability of throwing the ball when $n = 2$

   [SP2021, October]

3. A particle follows a random walk on the set described in the following figure



   At time n, the particle moves to one of the adjacent nodes with equal probability

   (a) Write the transition matrix of the chain

   (b) Find the invariant distribution

   (c) Is the chain irreducible and ergodic?

   [SP2021, July]

4. Consider exercise 2 on Markov chain and answer to point b hen n is large

5. Let X_n be a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

Suppose that the initial distribution is $\lambda = (1/2, 1/2, 0)$

(a) Find $P(X_0 = 1, X_2 = 3)$, $P(X_0 + X_2 = 4)$ and $P(X_0 = 1, X_2 = 3 | X_0 + X_2 = 4)$

(b) Is the chain ergodic? In case, find the invariant distribution

(c) Find

$$\lim_{n \to \infty} P(X_n + X_{n+2} = 4)$$

6. You have 2 urns, A and B. In the urn A there are 3 white balls, in the urn B there are 2 black balls. Let X_0 be the number of white balls in urn B, that is $X_0 = 0$. Suppose that you modify the urn composition in the following way. In each trial, you first take a ball from urn B and put the ball in the urn A, then you take a ball from urn A and you put the ball in urn B. Let X_1 be the number of white balls in urn B after the first complete switching and X_n be the number of white balls in urn B after n trials.

(a) Write the distribution of X_1

(b) Write the transition matrix P of the Markov Chain X_n

(c) Find the invariant distribution of the chain

[SP2021, June]

7. A cat and a mouse move independently back and forth between two rooms. At each time step, the cat moves from the current room to the other room with probability 0.8. Starting from room 1, the mouse moves to room 2 with probability 0.3 (and remains otherwise). Starting from room 2, the mouse moves to room 1 with probability 0.6 (and remains otherwise).

(a) Find the stationary distributions of the cat chain and of the mouse chain.

(b) Note that there are 4 possible (cat, mouse) states: both in room 1, cat in room 1 and mouse in room 2, cat in room 2 and mouse in room 1, and both in room 2. Number these cases 1, 2, 3, 4, respectively, and let Z_n be the number representing the (cat, mouse) state at time n. Explain why Z_n is still a Markov chain and find the transition matrix.

(c) Suppose that the cat and the mouse at time 0 are together in one of the two rooms with the same probability. What is the probability that they are in the same room at time 2?

(d) Now suppose that the cat will eat the mouse if they are in the same room. We wish to know the expected time (number of steps taken) until the cat eats the mouse for two initial configurations: when the cat starts in room 1 and the mouse starts in room 2, and vice versa. Set up a system of two linear equations in two unknowns whose solution is the desired values.

8. Each morning a student takes one of the three books (labelled 1, 2, 3) he owns from his shelf. The probability that he chooses the book with label i is α_i (where $0 < \alpha_i < 1$, i = 1, 2, 3), and choices on successive days are independent. In the evening he replaces the book at the left-hand end of the shelf. Let X_n be the order of the books at the end of day n.

   a Find the transition matrix of the chain $\{X_n; n = 1, 2 \ldots,\}$

   b Suppose that at time 0 the books are in the order 1,2,3, from left to right. What is the probability that at the end of day 2 the books are in the same order

   c If $p_{123}^{(n)}$ denotes the probability that on day n the student finds the books in the order 1,2,3, show that, irrespective of the initial arrangement of the books, $p_{123}^{(n)}$ converges as $n \to \infty$, and determine the limit.

   [SP2022, September]

Some solutions

1.

$$\boldsymbol{\pi}_1 = \lambda P = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$$

$$\begin{aligned}
\boldsymbol{\pi}_2 &= \boldsymbol{\pi}_1 P = \left(\frac{1}{16} + \frac{1}{6}, \frac{1}{8} + \frac{1}{6} + \frac{1}{8}, \frac{1}{16} + \frac{1}{6} + \frac{1}{8}\right) \\
&= \left(\frac{6+16}{96}, \frac{6+8+6}{48}, \frac{12+16+24}{192}\right) \\
&= \left(\frac{11}{48}, \frac{20}{48}, \frac{17}{48}\right)
\end{aligned}$$

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2. The transition matrix P is

$$\begin{array}{c} \\ A \\ B \\ C \end{array}
\begin{array}{c}
\begin{array}{ccc} A & B & C \end{array} \\
\begin{pmatrix} 0 & 0.2 & 0.8 \\ 0.6 & 0 & 0.4 \\ 0.5 & 0.5 & 0 \end{pmatrix}
\end{array}$$

$$\begin{aligned}
\lambda &= (1/3, 1/3, 1/3) \\
\boldsymbol{\pi}_1 &= \lambda P = (1/3)(1.1, 0.7, 1.2) \\
\boldsymbol{\pi}_2 &= \boldsymbol{\pi}_1 P = (1/3)(1.02, 0.82, 1.16) = (0.34, 0.273, 0.387)
\end{aligned}$$

The children with the higher probability is C

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3. The transition matrix is

$$P = \begin{pmatrix}
 & A & B & C & D & E \\
A & 0 & 1/2 & 1/2 & 0 & 0 \\
B & 1/2 & 0 & 1/2 & 0 & 0 \\
C & 1/4 & 1/4 & 0 & 1/4 & 1/4 \\
D & 0 & 0 & 1/2 & 0 & 1/2 \\
E & 0 & 0 & 1/2 & 1/2 & 0
\end{pmatrix}$$

The invariant distribution is the (row) vector $\pi = (\pi_A, \pi_B, \pi_C, \pi_D, \pi_E)$ such that $\pi P = \pi$ and $\sum_{j=A}^{E} \pi_j = 1$ which can be obtained by solving

$$\begin{cases}
\frac{1}{2}\pi_B + \frac{1}{4}\pi_C & = \pi_A \\
\frac{1}{2}\pi_A + \frac{1}{4}\pi_C & = \pi_B \\
\frac{1}{4}\pi_C + \frac{1}{2}\pi_E & = \pi_D \\
\frac{1}{4}\pi_C + \frac{1}{2}\pi_D & = \pi_E \\
\pi_A + \pi_B + \pi_C + \pi_D + \pi_E & = 1
\end{cases}$$

From the first two equations we obtain $\pi_B - \pi_A = 2(\pi_A - \pi_B)$, that is $\pi_A = \pi_B$ so that (look the first) $\pi_C = 2\pi_A$. From the third and fourth equations we obtain $\pi_E - \pi_D = 2(\pi_D - \pi_E)$, that is $\pi_D = \pi_E$ so that (look the third) $\pi_C = 2\pi_D$ which means $\pi_A = \pi_D$. Then from the last equation is $6\pi_A = 1$ and the invariant is $\pi = (1/6, 1/6, 1/3, 1/6, 1/6)$

Since all the states communicate the chain is irreducible. Irreducible finite chains have always the invariant distribution (which is unique) hence they are positive recurrent.

The chain is also aperiodic since $p_{AA}(2) > 0$ and $p_{AA}(3) > 0$ and the period is a class property.

Since the chain is ergodic and aperiodic the invariant distribution is also the limit distribution of the chain.

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4. Remember that the transition matrix is

$$P = \begin{pmatrix} 0 & 0.2 & 0.8 \\ 0.6 & 0 & 0.4 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

For this point we need to see if the chain is irreducible and aperiodic. In fact finite aperiodic and irreducible chains converge to the invariant distribution. The three states represent a communicating class, hence the chain is irreducible. The period is 1, in fact $p_{AA}(2) > 0$ and $p_{AA}(3) > 0$. Then the chain is irreducible and aperiodic.

The limit distribution is the invariant which solves the system

$$\begin{cases} 6\pi_B + 5\pi_C = 10\pi_A \\ 2\pi_A + 5\pi_C = 10\pi_B \\ \pi_A + \pi_B + \pi_C = 1 \end{cases}$$

From the first two equations we have $6\pi_B - 2\pi_A = 10(\pi_A - \pi_B)$ which is $\pi_A = \frac{4}{3}\pi_B$ and from the first we obtain $18\pi_B + 15\pi_C = 40\pi_B$ that is $\pi_B = \frac{15}{22}\pi_C$. The last equation is then $\pi_B(\frac{4}{3} + 1 + \frac{22}{15}) = 1$, that is $\pi_B = \frac{20+15+22}{15} = 1$, hence $\pi_B = \frac{15}{57}$, $\pi_A = \frac{20}{57}$ and $\pi_C = \frac{22}{57}$

In the long run (and approximately also when $n = 100$) C has the higher probability of throwing the ball

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5.

$$\begin{aligned} P(X_0 = 1, X_2 = 3) &= \sum_{j=1}^{3} P(X_0 = 1, X_1 = j, X_2 = 3) = P(X_0 = 1, X_1 = 2, X_2 = 3) \\ &= P(X_0 = 1)P(X_1 = 2|X_0 = 1)P(X_2 = 3|X_2 = 2) = \frac{1}{2}\frac{2}{3}\frac{2}{3} = \frac{2}{9} \end{aligned}$$

$$
\begin{aligned}
P(X_0+X_2=4) &= \sum_{j=1}^{3} P(X_0+X_2=4|X_0=j)P(X_0=j)\\
&= P(X_2=4-1|X_0=1)P(X_0=1)+P(X_2=4-2|X_0=2)P(X_0=2)\\
&= \frac{1}{2}\frac{2}{3}\frac{2}{3}+\frac{1}{2}\left(\frac{1}{3}\frac{2}{3}+\frac{2}{3}\frac{1}{3}\right)=\frac{2}{9}+\frac{2}{9}=\frac{4}{9}
\end{aligned}
$$

$$
P(X_0=1,X_2=3|X_0+X_2=4)=\frac{1}{2}
$$

It is ergodic and the invariant is $\pi=(1/3,1/3,1/3)$. For the invariant distribution note that the transition matrix is double stochastic, that is also the columns sum to 1. In this cases it easy to see that the uniform distribution is the invariant one.

Finally note that

$$
\begin{aligned}
P(X_n+X_{n+2}=4) &= \sum_{j=1}^{3} P(X_n+X_{n+2}=4|X_n=j)P(X_n=j)\\
&\approx \frac{1}{3}\left(\frac{4}{9}+\frac{4}{9}+\frac{1}{9}\right)=\frac{1}{3}
\end{aligned}
$$

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6. Note that at each time n the number of balls in urn B is always 2.

$$
P(X_1=0)=\frac{1}{4}
$$

In fact in the first trial, with probability 1 we put a black ball from B to A. Then in A we have 3 white and 1 black balls. To have 0 white balls in B at the end of the trial we need to select the black ball. This event occurs with probability $1/4$. Since $P(X_2=2)=0$ we have also

$$
P(X_1=1)=\frac{3}{4}
$$

That is $X_1\in\{0,1\}$ with $P(X_1=0)=1/4$ and $P(X_1=1)=3/4$

In general for this chain, $P(X_n=0|X_{n-1}=0)=1/4$, $P(X_n=1|X_n=0)=3/4$ and $P(X_n=2|X_n=0)=0$.

Now suppose that $X_n=1$, to have $X_{n+1}=0$ we need to select the white ball in the first part of the trial and a black in the second. Then

$$
P(X_{n+1}=0|X_n=1)=\frac{1}{2}\frac{1}{4}=\frac{1}{8}
$$

To have $X_{n+1}=1$ we can select the white ball from B to put in A and again a white ball in A to put in B, or a black ball in B to put in A and again a black from A to put in B. That is

$$
P(X_{n+1}=1|X_n=1)=\frac{1}{2}\frac{3}{4}+\frac{1}{2}\frac{1}{2}=\frac{5}{8}
$$

To have $X_{n+1} = 2$ we need to select the black ball in the first part of the trial and a white in the second. Hence, as expected,

$$P(X_{n+1} = 1|X_n = 1) = \frac{1}{2}\frac{1}{2} = \frac{2}{8}$$

Suppose that $X_n = 2$. Then we have 2 white balls in B. hence with probability 1 we take a white and A will have 2 white and 2 black balls. Hence

$$P(X_{n+1} = 1|X_n = 2) = P(X_{n+1} = 2|X_n = 2) = \frac{1}{2}$$

The transition matrix is then

$$P = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The invariant distribution satisfies the system

$$\begin{cases} \frac{1}{4}\pi_A + \frac{1}{8}\pi_B = \pi_A \\ \frac{2}{8}\pi_B + \frac{1}{2}\pi_C = \pi_C \\ \pi_A + \pi_B + \pi_C = 1 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} 2\pi_A + \pi_B = 8\pi_A \\ 2\pi_B + 4\pi_C = 8\pi_C \\ \pi_A + \pi_B + \pi_C = 1 \end{cases}$$

From the first equation we have $\pi_B = 6\pi_A$. From the second we have $\pi_C = \frac{1}{2}\pi_B = 3\pi_A$. The third equation is then $\pi_A + 6\pi_B + 3\pi_A = 1$ so that $\pi_A = \frac{1}{10}$. Hence the invariant distribution is

$$\pi = \left(\frac{1}{10}, \frac{6}{10}, \frac{3}{10}\right)$$

This is also the limit distribution of X_n since the chain is ergodic and aperiodic.

The transition matrices for the cat chain and the mouse chain are

$$\begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix} \quad \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix}$$

The stationary distribution for the cat chain solves the system

$$\begin{cases} \pi_1 0.2 + \pi_2 0.8 & = \pi_1 \\ \pi_1 + \pi_2 & = 1 \end{cases}$$

and it is $\pi = (1/2, 1/2)$

The stationary distribution for the mouse chain solves the system

$$\begin{cases} \pi_1 0.7 + \pi_2 0.6 & = \pi_1 \\ \pi_1 + \pi_2 & = 1 \end{cases}$$

and it is $\pi = (2/3, 1/3)$

For point b the transition matrix is

$$P = \begin{pmatrix} 0.14 & 0.06 & 0.56 & 0.24 \\ 0.12 & 0.08 & 0.48 & 0.32 \\ 0.56 & 0.24 & 0.14 & 0.06 \\ 0.48 & 0.32 & 0.12 & 0.08 \end{pmatrix}$$

Point c

$$\pi_0 = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right)$$

$$\pi_1 = \pi_0 P = \left(\frac{14+48}{200}, \frac{6+32}{200}, \frac{56+12}{200}, \frac{24+8}{200}\right) = \left(\frac{62}{200}, \frac{38}{200}, \frac{68}{200}, \frac{32}{200}\right)$$

$$\pi_2 = \pi_1 P = \left(\frac{6668}{20000}, \cdots, \cdots, \frac{3368}{20000}\right)$$

so that the probability that they are both in the same room at time 2 is 10036/20000

For the last point let t_1 be the expected time for the first configuration and t_2 be the expected time for the second configuration

The system is the following

$$t_1 = 0.12 \times 1 + 0.32 \times 1 + (1 + t_1) \times 0.08 + (1 + t_2) \times 0.48$$
$$t_2 = 0.56 \times 1 + 0.06 \times 1 + (1 + t_1) \times 0.24 + (1 + t_2) \times 0.14$$

The solution is $t_1 = 335/169$ $t_2 = 290/169$

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7. Transition matrix

stati	123	132	213	231	321	312
123	α_1	0	α_2	0	0	α_3
132	0	α_1	α_2	0	0	α_3
213	α_1	0	α_2	0	α_3	0
231	α_1	0	0	α_2	α_3	0
321	0	α_1	0	α_2	α_3	0
312	0	α_1	0	α_2	0	α_3

$$p_{123}^{(2)} = (\alpha_1, 0, \alpha_2, 0, 0, \alpha_3)'(\alpha_1, 0, \alpha_2, 0, 0, \alpha_3) = \alpha_1^2 + \alpha_1\alpha_2 = \alpha_1(1 - \alpha_3)$$

The chain has a finite number of states, it is irreducible and aperiodic. The limit distribution solves the system $\pi = \pi P$

$$
\begin{cases}
\pi_1 = (\pi_1 + \pi_3 + \pi_4)\alpha_1 \\
\pi_2 = (\pi_2 + \pi_5 + \pi_6)\alpha_1 \\
\pi_3 = (\pi_1 + \pi_2 + \pi_3)\alpha_2 \\
\pi_4 = (\pi_4 + \pi_5 + \pi_6)\alpha_2 \\
\pi_5 = (\pi_3 + \pi_4 + \pi_5)\alpha_3 \\
\pi_6 = 1 - \pi_1 - \pi_2 - \pi_3 - \pi_4 - \pi_5
\end{cases}
$$

Note that

$$
\begin{cases}
\alpha_1 = \pi_1 + \pi_2 \\
\alpha_2 = \pi_3 + \pi_4 \\
\alpha_3 = \pi_5 + \pi_6
\end{cases}
$$

Considering the first equation of the first system and the second equation from the second system we have

$$\pi_1 = \alpha_1\pi_1 + \alpha_1\alpha_2$$

and

$$\pi_1 = \frac{\alpha_1}{1 - \alpha_1}\alpha_2$$

- Recap about all

1. Let X and Y be independent Bernoulli random variables with the same parameter p. Consider $Z = X + Y$ and $V = X - Y$

   a Find the joint p.m.f. of Z, V, and the means $E(Z)$ and $E(V)$

   b Find the covariance $cov(Z, V)$

   c Are Z and V independent?

2. Let X and Y be independent Exponential random variables with $\lambda = 1$.

   (a) Find the distribution function and the density of the random variable $Z = Y/X$

   (b) Find the joint density of Z, U where $U = X + Y$ and find the marginal density of Z. Did you obtain the same result for $f_Z(z)$

   (c) Find the mean of Z, if it exists

3. At Christmas you received three boxes of chocolates as a gift. The first contains 20 milk chocolates, 12 dark chocolates and 8 filled with liqueur. The second contains 10 milk chocolates, 10 dark chocolates and 10 with liqueur. The third 10 milk and 10 fondant. To get a chocolate, choose the box by flipping a coin twice. If it comes up twice heads choose randomly the chocolate from the first box, if it comes out twice tails you choose it from the second, otherwise you choose it from the third. Once the box has been chosen, the chocolate is chosen at random.

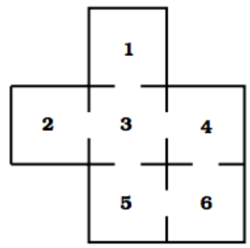
   (a) What is the probability that the first chocolate is a milk chocolate

   (b) You took a milk chocolate, what is the probability that you picked it from the first box

   (c) You have just eaten your first chocolate and it was milk. What is the probability that the second chocolate, taken from the same box, is also milk?

   (d) Now suppose that every time you take the candy, you put it back in the box anyway and don't eat it. Assuming to repeat the complete operation of tossing the coin and drawing the chocolate 5 times, what is the probability that at least one of the chocolates drawn is milk?

4. A rat runs through the maze shown below. At each step it leaves the room it is in by choosing at random one of the doors out of the room.



(a) Give the transition matrix P for this Markov chain.

(b) Show that it is irreducible but not aperiodic.

(c) Find the stationary distribution

(d) Now suppose that a piece of mature cheddar is placed on a deadly trap in Room 5. The mouse starts in Room 1. Find the expected number of steps before reaching Room 5 for the first time