ANALISI DELLE SERIE STORICHE
MODELLI GARCH

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Lezione 14

Modelli GARCH

GARCH models were introduced by Bollerslev 1986 as extensions of the ARCH models to allow for a much more flexible lag structure.

This flexibility is obtained by considering the variance of the error term as a function of both the previous time errors and variance of the process and thus, the GARCH(p,q)

$$\sigma_t^2 = \omega + \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2.$$

The process is weakly stationary if and only if

$$\sum_{j=1}^{p} \alpha_j + \sum_{j=1}^{q} \beta_j < 1,$$

GARCH models have become a widespread tool in time series analysis, especially for their usefulness in analyzing and forecasting in the presence of heteroskedasticity.

Moments of Gaussian GARCH(1,1)

We compute the first four conditional and unconditional moments of a GARCH(1,1) process, its auto-correlation function.

We consider the same structure defined before and so, for a time series r_t, a GARCH(1,1) model is defined as:

$$r_t = \epsilon_t$$

where

$$\epsilon_t | \mathscr{F}_{t-1} \sim N(0, \sigma_t^2)$$

and

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2. \quad (1)$$

The process is stationary when $\omega > 0$ and $\alpha + \beta < 1$. To ensure that the conditional variance is positive we have to impose that $\alpha \geq 0$ and $\beta \geq 0$

Moments of Gaussian GARCH(1,1)

- The first conditional and unconditional moments are both null

$$\mathbb{E}(r_t) = \mathbb{E}(\mathbb{E}(r_t|\mathscr{F}_{t-1})) = 0$$

- the second unconditional moment is:

$$\begin{aligned}\sigma^2 =&\mathbb{E}(r_t^2) = \mathrm{Var}(\epsilon_t) = \mathbb{E}(\epsilon_t^2) = \mathbb{E}(\mathbb{E}(\epsilon_t^2|\mathscr{F}_{t-1})) = \\ =&\mathbb{E}(\sigma_t^2) = \mathbb{E}(\omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2) = \omega + \alpha\mathbb{E}(\epsilon_{t-1}^2) + \beta\mathbb{E}(\sigma_{t-1}^2) = \\ =&\omega + \alpha\sigma^2 + \beta\sigma^2 \Rightarrow \sigma^2 = \frac{\omega}{1-\alpha-\beta} \iff \alpha + \beta < 1\end{aligned}$$

Moments of Gaussian GARCH(1,1)

The third unconditional moment of a stationary GARCH(1,1) model is:

$$\mathbb{E}(\epsilon_t^3) = \mathbb{E}(\mathbb{E}(\epsilon_t^3|\mathscr{F}_{t-1})) = 0.$$

The result derives from the result on the third moment of a Gaussian distribution. To evaluate the fourth unconditional moment we use the following result:

$$\begin{aligned}\mathbb{E}(\epsilon_t^4|\mathscr{F}_{t-1}) =& 3\mathbb{E}(\epsilon_t^2|\mathscr{F}_{t-1})^2 \\ &= 3Var(\epsilon_t|\mathscr{F}_{t-1})^2 \\ &= 3(\sigma_t^2)^2 \\ &= 3\sigma_t^4.\end{aligned}$$

The fourth unconditional moment of a stationary GARCH(1,1) model is:

$$
\begin{aligned}
\mu_4 &= \mathbb{E}(\epsilon_t^4) = \mathbb{E}(\mathbb{E}(\epsilon_t^4|\mathscr{F}_{t-1})) = 3\mathbb{E}(\sigma_t^4) = 3\mathbb{E}(\omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2)^2 \\
&= 3\omega^2 + 3\alpha^2\mathbb{E}(\epsilon_{t-1}^4) + 3\beta^2\mathbb{E}(\sigma_{t-1}^4) + \\
&+ 6\omega\alpha\mathbb{E}(\epsilon_{t-1}^2) + 6\omega\beta\mathbb{E}(\sigma_{t-1}^2) + 6\alpha\beta\mathbb{E}(\epsilon_{t-1}^2\sigma_{t-1}^2)
\end{aligned}
$$

Now considering that

$$
\begin{aligned}
\mathbb{E}(\epsilon_{t-1}^2\sigma_{t-1}^2) &= \mathbb{E}(\mathbb{E}(\epsilon_{t-1}^2\sigma_{t-1}^2|\mathscr{F}_{t-2})) = \\
&= \mathbb{E}(\sigma_{t-1}^2\mathbb{E}(\epsilon_{t-1}^2|\mathscr{F}_{t-2})) = \\
&= \mathbb{E}(\sigma_{t-1}^2\sigma_{t-1}^2) = \mathbb{E}(\sigma_{t-1}^4)
\end{aligned}
$$

Since from the previous result we have that

$$\mathbb{E}(\sigma_t^4) = \frac{\mathbb{E}(\epsilon_t^4)}{3}$$

we get

$$\mu_4 = 3\omega^2 + 3\alpha^2\mu_4 + \beta^2\mu_4 + 6\omega\sigma^2 + 6\omega\beta\sigma^2 + 6\alpha\beta\mu_4/3$$

then

$$\mu_4(1 - 3\alpha^2 - \beta^2 - 2\alpha\beta) = 3\omega^2 + 6\omega\sigma^2(\alpha + \beta)$$

and

$$\mu_4 = \frac{3\omega^2(1 + \alpha + \beta)}{(1 - \alpha - \beta)(1 - 3\alpha^2 - \beta^2 - 2\alpha\beta)}$$

we can calculate the kurtosis index as

$$Kurt = \frac{\mu_4}{\mathbb{E}(\epsilon_t^2)} = \frac{3\omega^2(1+\alpha+\beta)}{(1-\alpha-\beta)(1-3\alpha^2-\beta^2-2\alpha\beta)} \frac{(1-\alpha-\beta)^2}{\omega^2} =$$
$$= \frac{3(1-\alpha-\beta)(1+\alpha+\beta)}{(1-3\alpha^2-\beta^2-2\alpha\beta)} > 3$$

It is possible to show that GARCH(1,1) model can be written as an ARMA(1,1) process

$$r_t^2 = \omega + (\alpha + \beta) r_{t-1}^2 - \beta v_{t-1} + v_t$$

where v_t is a WN with

$$\mathbb{E}(v_t) = 0$$

$$Var(v_t) = \frac{2(1 + \alpha + \beta)}{(1 - \alpha - \beta)(1 - 3\alpha^2 - \beta^2 - 2\alpha\beta)}$$

$$Cov(v_t, v_{t-h}) = 0$$

we now calculate the $Cov(\epsilon_t^2, \epsilon_{t-h}^2)$ starting from the ARMA(1,1) representation of a GARCH(1,1) process.
It is worth noting that the stationary condition of the GARCH process i.e. $\alpha + \beta < 1$ coincide with the stationary condition of its ARMA(1,1) representation. So since the $\gamma(h)$ for an ARMA(1,1) process with autoregression parameter ϕ, moving average parameter θ and variance of the error terms σ^2 is:

$$\gamma(h) = \mathbb{C}\mathrm{ov}(\epsilon_t^2, \epsilon_{t-h}^2) = \begin{cases} \dfrac{1 + 2\phi\theta + \theta^2}{1 - \phi^2}\sigma^2 & h = 0 \\ \phi^{h-1}\dfrac{(\phi + \theta)(1 + \phi\theta)}{1 - \phi^2}\sigma^2 & h > 0 \end{cases}$$

considering that in our representation we have:

- $\phi = \alpha + \beta$;
- $\theta = -\beta$;
- $\sigma^2 = \mathrm{Var}(v_t)$.

after some algebra we can write that

$$\gamma(h) = \mathbb{C}\mathrm{ov}(\epsilon_t^2, \epsilon_{t-h}^2) = \frac{2\alpha\omega^2(1-\alpha\beta-\beta^2)}{(1-\alpha-\beta)^2(1-3\alpha^2-\beta^2-2\alpha\beta)}(\alpha+\beta)^{h-1}.$$

to compute the ACF we have to calculate the $\mathrm{Var}(\epsilon_t^2) = \mathbb{E}(\epsilon_t^4) - \mathbb{E}(\epsilon_t^2)$. From previous results we have

$$\mathrm{Var}(\epsilon_t^2) = \frac{3\omega^2(1+\alpha+\beta)}{(1-\alpha-\beta)(1-3\alpha^2-\beta^2-2\alpha\beta)} - \left(\frac{\omega}{1-\alpha-\beta}\right)^2$$

which after some algebra can be written as:

$$\mathrm{Var}(\epsilon_t^2) = \frac{2\omega^2(1-2\alpha\beta-\beta^2)}{(1-\alpha-\beta)^2(1-3\alpha^2-\beta^2-2\alpha\beta)}.$$

Substituting this results in the ACF definition we finally have

$$\rho(h) = \frac{\mathbb{C}\mathrm{ov}(\epsilon_t^2, \epsilon_{t-h}^2)}{\mathrm{Var}(\epsilon_t^2)} = \frac{\frac{2\alpha\omega^2(1-\alpha\beta-\beta^2)}{(1-\alpha-\beta)^2(1-3\alpha^2-\beta^2-2\alpha\beta)}(\alpha+\beta)^{h-1}}{\frac{2\omega^2(1-2\alpha\beta-\beta^2)}{(1-\alpha-\beta)^2(1-3\alpha^2-\beta^2-2\alpha\beta)}}$$

and simplifying it leads to

$$\rho(h) = \frac{\alpha(1-\alpha\beta-\beta^2)}{1-2\alpha\beta-\beta^2}(\alpha+\beta)^{h-1}.$$

Extensions

Classical GARCH models, rely on modeling the conditional variance as a linear function of the squared past innovations. The merits of this specification are its ability to reproduce several important characteristics of financial time series.
From an empirical point of view, however, the classical GARCH modeling has an important drawback. Indeed, by construction, the conditional variance only depends on the modulus of the past variables: past positive and negative innovations have the same effect on the current volatility. This property is in contradiction to many empirical studies on series of stocks, showing a negative correlation between the squared current innovation and the past innovations. However, conditional asymmetry is a stylized fact: the volatility increase due to a price decrease is generally stronger than that resulting from a price increase of the same magnitude In order to take into account for those considerations some extensions of GARCH model have been considered in literature.

EGARCH(p,q)

$$r_t = \epsilon_t$$

where

$$\epsilon_t | \mathscr{F}_{t-1} i.i.d(0, \sigma_t^2)$$

and

$$\log(\sigma_t^2) = \omega + \sum_{i=1}^{n} \alpha_i g(\epsilon_{t-i}) + \sum_{i=1}^{n} \beta_j \log(\sigma_{t-j}^2).$$

where

$$g(\epsilon_{t-i}) = \theta\epsilon_{t-i} + \gamma(|\epsilon_{t-i}| - \mathbb{E}(|\epsilon_{t-i}|)$$

To better understand the meaning of the function $g(\dot{})$ we can write it as

$$g(\epsilon_t) = \begin{cases} (\theta + \gamma)\epsilon_t - \gamma\mathbb{E}(|\epsilon_t|) & \epsilon_t \geq 0 \\ (\theta - \gamma)\epsilon_t - \gamma\mathbb{E}(|\epsilon_t|) & \epsilon_t \geq 0 \end{cases}$$

which means that for positive ϵ_t $g(\epsilon_t)$ is a linear function of ϵ_t with angular coefficient $\theta + \gamma$ while for negative ϵ_t is a linear function of ϵ_t with angular coefficient $\theta - \gamma$. In this way the model reacts in an asymmetric way to positive and negative news.
Moreover it is worth noting that with the EGARCH specification we do not need to impose any restrictions on the parameters to constraint the variance to be positive. In fact the exponential of any number is always positive.
when $\theta = 0$ the reaction of $\log(\sigma_t^2)$ to a variation of ϵ_t is symmetric

TGARCH(p,q)

This model is called Treshold GARCH

$$r_t = \epsilon_t$$

where

$$\epsilon_t | \mathscr{F}_{t-1} i.i.d(0, \sigma_t^2)$$

and

$$\sigma_t^2 = \omega + \sum_{i=1}^{n} \alpha_i \epsilon_{t-i}^2 + \gamma_i \epsilon_{t-i}^2 I_{(-\infty,0)}(\epsilon_{t-i}) \sum_{i=j}^{n} \beta_j \log(\sigma_{t-j}^2).$$

where $I_A(x)$ is the indicator function of the set A and the parameters are non negative.

The model is asymmetric since when ϵ_{t-i} is positive it contributes with α_i to the volatility while when it is negative it contributes with $\alpha_i + \gamma_i$ which has a larger impact.
The model use zero as threshold to separate the impacts of past shocks. Other threshold values can also be used