Stochastic Processes

Andrea Tancredi

Sapienza University of Rome

Week 3

Examples of discrete random variables

Functions of discrete random variables

Expectation

Variance

- **Geometric**

  We say that X has the Geometric distribution (X is a Geometric r.v.) with parameter $p \in (0,1)$ if X takes value $\{1,2,\ldots\}$ and

$$P(X=k) = p\,q^{k-1} \quad \text{for } k = 1,2,\ldots$$

  Observe that for $x \in \mathbb{R}$

$$\begin{aligned}(1-x)\sum_{k=0}^{n-1} x^k &= \sum_{k=0}^{n-1} x^k - \sum_{k=1}^{n} x^k \\ &= 1 + x + x^2 + \cdots x^{n-1} - (x + x^2 + \cdots x^n) = 1 - x^n\end{aligned}$$

  Then

$$\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$$

  and

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=0}^{n-1} x^k = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \begin{cases} \frac{1}{1-x} & |x| < 1 \\ \infty & |x| \geq 1 \end{cases}$$

Hence we have also that the sum of the geometric probabilities is 1

$$\sum_{k=1}^{\infty} pq^{k-1} = p \sum_{k=0}^{\infty} q^k = p \frac{1}{1-q} = \frac{p}{p} = 1$$

Exercise 1.17 Let $p_1, p_2, \dots p_N$ non negative numbers such that $\sum_{i=1}^N p_i = 1$. Consider $\Omega = \{\omega_1, \dots, \omega_N\}$ and let $\mathcal{F}$ be the power set of Ω

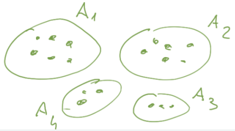
$$Q(A) = \sum_{i:\omega_i \in A} p_i$$

is a probability measure

- $\varphi(\Omega) = 1$. In fact $\varphi(\Omega) = \sum_{i:\omega_i \in \Omega} p_i = \sum_{i=1}^{N} p_i = \underline{1}$
- $\varphi(A) \geq 0$ $\forall A \in \mathcal{F}$. In fact $\varphi(A) = \sum_{i:\omega_i \in \Omega} p_i \geq 0$ since $p_i \geq 0$ $\forall i$
- Let $A = \bigcup_{j \in J} A_j$ where $A_j \cap A_{j'} = \emptyset$ $j \neq j'$

$$P(A) = P\Big(\bigcup_{j \in J} A_j\Big) = \sum_{i:\omega_i \in \bigcup_{j \in J} A_j} p_i = \sum_{j \in J} \sum_{i:\omega_i \in A_j} p_i$$

$$= \sum_{j \in J} P(A_j)$$



Example 2.18, A coin is tossed n times. $|\Omega| = 2^n$, sngle outcomes are

$$\underbrace{TTHT \cdots TTH}_{n \text{ coins}}$$

- Let $h(\omega)$ be the number of heads and $t(\omega)$ be the number of tails, $t(\omega) = n - h(\omega)$. Consider $p \in (0,1)$ and $q = 1 - p$. Let $\mathcal{F}$ be the power set
- Take

$$p(\omega) = p^{h(\omega)} q^{t(\omega)}$$

  and

$$P(A) = \sum_{\omega \in A} p(\omega)$$

Note that

$$\begin{aligned}
\sum_{\omega\in\Omega} p(\omega) &= \sum_{k=0}^{n} \sum_{\omega:h(\omega)=k} p(\omega) = \sum_{k=0}^{n} \sum_{\omega:h(\omega)=k} p^{h(\omega)} q^{t(\omega)} \\
&= \sum_{k=0}^{n} \sum_{\omega:h(\omega)=k} p^k q^{n-k} = \sum_{k=0}^{n} p^k q^{n-k} \sum_{\omega:h(\omega)=k} 1 \\
&= \sum_{k=0}^{n} p^k q^{n-k} \binom{n}{k} = 1
\end{aligned}$$

Then P is a probability (by ex.1.17)

Let

$$X_i(\omega) = \begin{cases} 1 & \text{if the } i\text{th entry in } \omega \text{ is } H \\ 0 & \text{if the } i\text{th entry in } \omega \text{ is } T \end{cases}$$

$$\begin{aligned}
P(X_i = 0) &= P(\omega \in \Omega : \omega_i = T) = \sum_{\omega:\omega_i = T} p(\omega) = \sum_{\omega:\omega_i = T} p^{h(\omega)} q^{t(\omega)} \\
&= \sum_{h=0}^{n-1} \sum_{\omega:h(\omega)=h,\omega_i=T} p^{h(\omega)} q^{t(\omega)} = \sum_{h=0}^{n-1} \sum_{\omega:h(\omega)=h,\omega_i=T} p^h q^{n-h} \\
&= \sum_{h=0}^{n-1} p^h q^{n-h} \sum_{\omega:h(\omega)=h,\omega_i=T} 1 = \sum_{h=0}^{n-1} p^h q^{n-h} \binom{n-1}{h} \\
&= \mathbf{q} \underbrace{\sum_{h=0}^{n-1} \binom{n-1}{h} p^h q^{n-1-h}}_{1} = \mathbf{q}
\end{aligned}$$

Hence, each X_i takes the values 0 and 1 and $P(X_i = 0) = q$ and $P(X_i = 1) = p$, that is

$$X_i \sim Bernoulli(p) \quad i = 1, \ldots n$$

Example 2.18 (a coin is tossed n times)

$\Omega = \{HHH\dots H, THH\dots H, \dots, TTT\dots T\}$ $\omega = (\omega_1 \omega_2 \dots \omega_n)$ where ω_i can be H or T

$P(\omega) = p^{h(\omega)} (1-p)^{t(\omega)}$

We have seen that $X_i = X_i(\omega) = \begin{cases} 1 & \text{if } \omega_i = H \\ 0 & \text{if } \omega_i = T \end{cases}$

is a Bernoulli r.v. with parameter p

Consider $S_n = X_1 + X_2 \dots + X_n$

$$S_n : \Omega \to \{0, 1 \dots n\}$$

in fact $S_n = S_n(\omega) = X_1(\omega) + X_2(\omega) + \dots \; X_n(\omega)$

Remember that $P(A) = \sum_{\omega \in A} P(\omega) = \sum_{\omega \in A} p^{h(\omega)} (1-p)^{t(\omega)}$

$$P(S_n = k) = P(\omega \in \Omega : S_n(\omega) = k) = P(\omega \in \Omega : h(\omega) = k) =$$

$$= \sum_{\omega : h(\omega) = k} p^{h(\omega)} (1-p)^{t(\omega)} = \sum_{\omega : h(\omega) = k} p^k (1-p)^{n-k} = \binom{n}{k} p^k q^{n-k}$$

Hence S_n is Binomial with parameters n and p

Poisson distribution as a limit of Binomial

Suppose that n is very large, p is very small and $\lambda = n \cdot p$ does not go to zero or to infinity

$$P(S_n = k) = \binom{n}{k} p^k q^{n-k} = \frac{n!}{(n-k)!\,k!} p^k q^{n-k}$$

$$\sim \underbrace{\frac{n^k}{k!}\left(\frac{\lambda}{n}\right)^k}_{\sim \frac{\lambda^k}{k!}} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\sim e^{-\lambda}} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{\sim 1} \sim \frac{\lambda^k}{k!} e^{-\lambda}$$

Example 2.21 Geometric distribution

Consider the following experiment: we toss a coin until the first head turns up and then we stop

$$\Omega = \{H, TH, TTH, TTTH, \cdots\} \bigcup \{TTTT\cdots\}$$

Let $\omega = T^k H$ be the outcome with k heads and assume that

$$p(\omega) = q^k p \quad p \in (0,1) \;\; q = 1 - p$$

For the outcome $\omega = T^\infty$ without heads we take

$$p(\omega) = \begin{cases} 1 & p = 0 \\ 0 & p > 0 \end{cases}$$

Note that if $p > 0$

$$\sum_{\omega\in\Omega} p(\omega) = \sum_{k=0}^{\infty} p(T^k H) + p(T^\infty) = \sum_{k=0}^{\infty} pq^k + 0 = p\frac{1}{1-q} = 1$$

If $p = 0$ we have also $\sum_{\omega\in\Omega} p(\omega) = 0 + 0 + 0 + \cdots + 1 = 1$

Consider the application

$$Y : \Omega \to \mathbb{R}$$

where

$$Y(\omega) = Y(T^k H) = k + 1$$

Then

$$Y \in \{1, 2, \dots\}$$

and

$$P(Y = k) = P(T^{k-1} H) = q^{k-1} p \quad k = 1, 2, \dots$$

Hence Y has geometric distribution with parameter p

Exercise 2.24. Show that if Y is Geometric p, $P(Y > k) = (1 - p)^k$

Functions of discrete random variables

Let X be a discrete random variable on $(\Omega, \mathcal{F}, P)$ and let $g : \mathbb{R} \to \mathbb{R}$.
Then $Y = g(X)$ defined as

$$Y(\omega) = g(X(\omega))$$

is a random variable on $(\Omega, \mathcal{F}, P)$
Some examples

- if $g(x) = ax + b$ then $g(X) = aX + b$
- if $g(x) = cx^2$ then $g(X) = cX^2$
- if $g(x) = \exp(x)$ then $g(X) = \exp(X)$

If $Y = g(X)$ the probability mass function of Y is

$$P_Y(y) = \sum_{x \in g^{-1}(y)} P(X = x)$$

where $g^{-1}(y) = \{x : g(x) = y\}$. In fact

$$\{Y = y\} = \{g(X) = y\} = \{X \in \{x : g(x) = y\}\} = \{X \in g^{-1}(y)\}$$

Then $P_Y(y) = P(g(X) = y) = P(X \in \{x : g(x) = y\}) =$

$$= \sum_{x : g(x) = y} P(X = x) = \sum_{x \in g^{-1}(y)} P(X = x)$$

- If $Y = aX + b$ then

$$P(Y = y) = P(aX + b = y) = P(X = (y - b)/a)$$

- If $Y = X^2$ for $y > 0$

$$P(Y = y) = P(X^2 = y) = P(X = -\sqrt{y}) + P(X = \sqrt{y})$$

Moreover for $y = 0$

$$P(Y = 0) = P(X^2 = 0) = P(X = 0)$$

and for $y < 0$

$$P(Y = y) = 0$$

Expectation

Definition. If X is a discrete r.v. the **expectation** (expected value, mean) of X is denoted with $E(X)$ and defined by

$$E(X) = \sum_{x \in Im(X)} x\, P(X = x)$$

whenever this sum converges absolutely in that $\sum_x |xP(X = x)| < \infty$

Expecation for some important discrete random variables X

- Bernoulli(p), $E(X) = p$
- Binomial(n,p,) $E(X) = np$
- Poisson(λ), $E(X) = \lambda$
- Geometric(p), $E(X) = 1/p$
- Discrete uniform on $\{0, \ldots n\}$, $E(X) = n/2$

$X \sim$Bernoulli(p)

$$E(X) = \sum_x x\, P(X = x) = 0 \cdot (1 - p) + 1 \cdot p = p$$

$X \sim$ Poisson(λ)

$$\begin{aligned} E(X) &= \sum_x x\, P(X = x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{(x-1)}}{(x-1)!} \\ &= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} = \lambda \end{aligned}$$

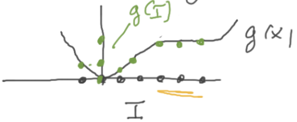
THEOREM: If X is a discrete random variable and $g: \mathbb{R} \to \mathbb{R}$ then $E(g(X)) = \sum_{x \in \mathrm{Im}(X)} g(x) P(X=x)$ whenever this sum converges absolutely.

(In fact $Y = g(X)$ takes the value $g(x)$ when X takes the value x, and this event has probability $P(X=x)$

Each value $g(x)$ [*] receives the "weight $P(X=x)$

Proof

Let I the image of X. The image of Y is $g(I)$



$$E(Y) = \sum_{y \in g(I)} y\, P(Y=y) = \sum_{y \in g(I)} y \cdot \sum_{x : g(x) = y} P(X=x)$$

$x \in g^{-1}(y)$

$$= \sum_{y \in g(I)} \sum_{x : g(x) = y} y \cdot P(X=x) = \sum_{y \in g(I)} \sum_{x : g(x) = y} g(x)\, P(X=x)$$

$$= \sum_{x \in I} g(x) \cdot P(X=x)$$

THEOREM

Let X be a discrete random variable. If $P(X \geq 0) = 1$ (X takes non negative values) and $E(X) = 0$, then $P(X=0) = 1$

PROOF

Infact $0 = \sum_{x} x P(X=x) = 0 \cdot P(X=0) + \sum_{x>0} x \cdot P(X=x)$

Hence $\sum_{x>0} x P(X=x) = 0$ which is possible only if $P(X=x) = 0 \quad \forall x > 0$. Therefore $P(X=0) = 1$

Exercise

Suppose that X is a discrete random variable with mean $\mu = E(X)$. Consider $Y = aX + b$. Find $E(Y)$

$Y = g(X)$ where $g = ax + b$

$$E(Y) = \sum_x g(x) P(X=x) = \sum_x (ax+b) P(X=x) =$$

$$= \sum_x ax\, P(X=x) + \sum_x b \cdot P(X=x) = a \cdot \sum_x x\, P(X=x) + b \sum_x P(X=x) =$$

$$= a E(X) + b = a\mu + b$$

Variance

Definition The variance $\mathrm{var}(X)$ of a discrete random variable X is defined by

$$\mathrm{var}(X) = E([X - E(X)]^2)$$

Consider $g(x) = (x - \mu)^2$ where $\mu = E(X)$ and remember that

$$E(g(X)) = \sum_{x \in Im(X)} g(x) P(X = x)$$

Then

$$\mathrm{var}(X) = \sum_{x \in Im(X)} (x - \mu)^2 P(X = x)$$

Note that $\text{var}(X) = E(X^2) - E(X)^2$

$$\begin{aligned}
\text{var}(X) &= \sum_x (x-\mu)^2 P(X=x) \\
&= \sum_x (x^2 - 2x\mu + \mu^2) P(X=x) \\
&= \sum_x x^2 P(X=x) - 2\mu \sum_x x P(X=x) + \mu^2 \sum_x P(X=x) \\
&= E(X^2) - 2\mu\mu + \mu^2 = E(X^2) - \mu^2 = E(X^2) - E(X)^2
\end{aligned}$$

Consider $Y = aX$, then $var(Y) = a^2 var(X)$

Infact $E(Y) = E(aX) = aE(X)$ and

$$var(Y) = E(Y^2) - E(Y)^2 = E(a^2 X^2) - (aE(X))^2$$

$$= a^2 E(X^2) - a^2 E(X)^2 =$$

$$= a^2 (E(X^2) - E(X)^2) = a^2 var(X)$$

Exercise: Consider $Y = aX + b$, show that $var(Y) = a^2 var(X)$

Variance for some important discrete random variables X

- Bernoulli(p), $\text{var}(X) = pq$
- Binomial(n,p,) $\text{var}(X) = npq$
- Poisson(λ), $\text{var}(X) = \lambda$
- Geometric(p), $var(X) = q/p^2$