Stochastic Processes

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Week 5

Generating functions

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Generating functions

Given the sequence

$$u_0, u_1, u_2, \ldots, u_n, \ldots$$

we can define the power series

$$u_0 + u_1 s + u_2 s^2 + \cdots u_n s^n + \cdots = \sum_{n=0}^{\infty} u_n s^n$$

for the values s such that $U(s) = \sum_{n=0}^{\infty} u_n s^n < \infty$

Examples

- Consider the sequence 2^n for $n = 0, 1, \ldots$

  $$1, 2, 4, 8, 16, 24, \ldots$$

  then

  $$1 + 2s + 4s^2 + 8s^3 + \cdots = \sum_{n=0}^{\infty} 2^n s^n = \sum_{n=0}^{\infty} (2s)^n = \frac{1}{1 - 2s} \quad |2s| < 1$$

- Consider the sequence $n + 1$ for $n = 0, 1, \ldots$

$$1, 2, 3, 4, \ldots$$

then

$$\begin{aligned}
1 + 2s + 3s^2 + 4s^3 + \cdots &= \sum_{n=0}^{\infty} (n+1) s^n \\
&= \sum_{n=0}^{\infty} \frac{d}{ds} s^{n+1} \\
&= \frac{d}{ds} \sum_{n=0}^{\infty} s^{n+1} = \frac{d}{ds} s \sum_{n=0}^{\infty} s^n \\
&= \frac{d}{ds} \frac{s}{1-s} = \frac{(1-s)+s}{(1-s)^2} = \frac{1}{(1-s)^2} \quad |s| < 1
\end{aligned}$$

- Consider the sequence $u_n = \binom{N}{n}$ for $n = 0, 1, \ldots, N$, and $u_n = 0$ for $n = N + 1, N * 2, \ldots$ Then

$$\begin{aligned} \binom{N}{0} + \binom{N}{1}s + \binom{N}{2}s^2 + \cdots + \binom{N}{N}s^N &= \sum_{n=0}^{N} \binom{N}{n} s^n \\ &= \sum_{n=0}^{N} \binom{N}{n} s^n 1^{N-n} \\ &= (1+s)^N \end{aligned}$$

Given a sequence u_n $n = 0, 1, \ldots$ we define generating function of the sequence the function

$$U(s) = u_0 + u_1 s + u_2 s^2 + \cdots = \sum_{n=0}^{\infty} u_n s_n$$

for the value $s : \sum_{n=0}^{\infty} u_n s^n < \infty$. Note that $U(0) = u_0$

A generating function specifies the sequence uniquely. In fact given the generating function $U(s)$ suppose that

$$U(s) = \sum_{n=0}^{\infty} u_n s^n \quad \text{and} \quad U(s) = \sum_{n=0}^{\infty} v_n s^n$$

for all the values s in a neighborhood of 0, $I(0)$. Then

$$u_n = v_n \quad n = 1, 2 \ldots$$

To prove the last sentence observe that for $s \in I(0)$

$$U(s) = u_0 + u_1 s + u_2 s^2 + \cdots = v_0 + v_1 s^1 + v_2 s^2 + \cdots = U(s)$$

By taking $s = 0$ we have $u_0 = v_0$ hence

$$u_1 s + u_2 s^2 + \cdots = v_1 s^1 + v_2 s^2 + \cdots$$

Now consider $s \neq 0$, $s \in I(0)$ and divide both the new series by s

$$u_1 + u_2 s + u_3 s^2 \cdots = v_1 + v_2 s + v_3 s^2 + \cdots$$

By taking $s = 0$ we have $u_1 = v_1$. Note also that the orange and purple series converge. ...Then repeat...

The coefficients u_n are also those of the Taylor series of $U(s)$ in 0, i.e. $u_n = \frac{U^{(n)}(0)}{n!}$ where where $U^{(n)}(a)$ denotes the nth derivative of U evaluated at the point a.

Integer-valued random variables

Consider a random variable X that takes value in the set of non-negative integers. We may think of the mass function of X as a sequence

$$p_0,\ p_1,\ p_2,\ \ldots$$

where

$$p_k = P(X = k) \quad \text{for } k = 0, 1, 2$$

satisfying

$$p_k \geq 0 \quad \text{for all } k, \quad \text{and} \quad \sum_{k=0}^{\infty} p_k = 1$$

Definition The probability generating function of X is the generating function of the sequence p_0, p_1, p_2, that is the function

$$G_X(s) = p_0 + p_1 s + p_2 s^2 + \cdots = \sum_{k=0}^{\infty} p_k s^k$$

for all the values of s for which $\sum_{k=0}^{\infty} |p_k s^k| < \infty$

Note that

- $G_X(0) = p_0 + p_1 0 + p_2 0 + \cdots = p_0$
- $G_X(1) = p_0 + p_1 1 + p_2 1 + \cdots = 1$
- $G_X(s) = \sum_{k=0}^{\infty} p_k s^k = E(s^X)$
- $G_X(s)$ exist fo all values of $s : |s| \leq 1$, since for these values

$$\sum_{k=0}^{\infty} |p_k s^k| \leq \sum_{k=0}^{\infty} p_k = 1$$

Probability generating function of the geometric distribution

$X \sim Geometric(p)$ where $P(X=k) = q^{k-1}p$ where $k = 1, 2 \dots$

$$G_X(s) = E(s^X) = \sum_{k=0}^{\infty} s^k P(X=k) = \sum_{k=1}^{\infty} s^k q^{k-1} p = p \cdot s \sum_{k=1}^{\infty} (s q)^{k-1} =$$

$$= p \cdot s \frac{1}{1-qs} \quad \text{for } |qs| < 1 \iff |s| < q^{-1}$$

Theorem: Uniqueness theorem for probability generating functions
Suppose X and Y have probability generating funtions G_X and G_Y, respectively. Then

$$G_X(s) = G_Y(s) \quad \text{for all } s$$

if and only if

$$P(X = k) = P(Y = k) \quad \text{for } k = 0, 1, 2$$

Proof if. If $P(X = k) = P(Y = k)$ for all k then (by definition of p.g.f.) $G_X(s) = G_Y(s)$

Proof only if. Suppose that $G_X(s) = G_Y(s)$. G_X and G_Y are both finite for $|s| \leq 1$ and cannot have different coefficients since they are power series, that is

$$P(X = k) = P(Y = k) \quad k = 0, 1, \ldots$$

The previous theorem is very important: if we discover that the p.g.f of X is that of a Geometric we can say that X is Geometric, if the p.g.f of X is that of a Binomial we can say that X is Binomial....

Examples of probability generating functions

$X \sim \text{Bernoulli}(p)$

$$E(s^X) = s^0(1-p) + s^1 \cdot p = (1-p) + ps$$

$X \sim \text{Binomial}(n, p)$

$$E(s^X) = \sum_{k=0}^{n} s^k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (ps)^k (1-p)^{n-k} = \left((1-p) + ps\right)^n$$

Definition let $k \geq 1$. The k*th* moment of X is the quantity $E(X^k)$
Remember that

$$\begin{aligned} var(X) &= E((X - E(X))^2) = E(X^2 - 2XE(X) + E(X)^2) \\ &= E(X^2) - 2E(X)E(X) + E(X)^2 \\ &= E(X^2) - E(X)^2 \\ &= \text{second moment of } X - (\text{first moment of } X)^2 \end{aligned}$$

We have also that

$$E(X^2) = E(X^2 - X + X) = E(X(X-1)) + E(X)$$

Theorem Let X be an integer valued random variable with probability generating function $G_X(s)$. The rth derivative of $G_X(s)$ at $s = 1$ equals $E(X(X-1)\cdots(X-(r-1))$, that is

$$G_X^{(r)}(1) = E(X(X-1)\cdots(X-(r-1)))$$

Proof. When $r = 1$

$$G_X'(s) = \frac{d}{ds} G_X(s) = \frac{d}{ds} \sum_{k=0}^{\infty} P(X=k) \cdot s^k = \sum_{k=0}^{\infty} \frac{d}{ds} s^k P(X=k) = \sum_{k=0}^{\infty} k s^{k-1} P(X=k)$$

Hence $G_X'(1) = \sum_{k=0}^{\infty} k \cdot P(X=k) = E(X)$

- We will use the theorem mainly to calculate mean and variance

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 = E(X(X-1)) + E(X) - E(X)^2 \\ &= G_X''(1) + G_X'(1) - G_X'(1)^2 \end{aligned}$$

Sums of independent random variables

Theorem If X and Y are independent integer-valued random variables with probability generating functions $G_X(s)$ and $G_Y(s)$ then $X+Y$ has probability generating function

$$G_{X+Y}(s) = G_X(s)G_Y(s)$$

Proof

$$G_{X+Y}(s) = E\left(s^{(X+Y)}\right) = E(s^X \cdot s^Y) = E(s^X) \cdot E(s^Y)$$

$$= G_X(s) \cdot G_Y(s)$$

Sum of n random variables

Given $(\Omega, \mathcal{F}, P)$ we may also have a multivariate random variable $(X_1, X_2 \dots X_n)$ also said random vector.

Important facts

$P(X_1 = x_1, X_2 = x_2, \dots X_n = x_n) = p_{X_1 \dots X_n}(x_1, x_2 \dots x_n)$ is the joint p.m.f of $X_1 \dots X_n$

$$E(X_1 + X_2 + \dots X_n) = E(X_1) + E(X_2) + \dots E(X_n)$$

If $\forall (x_1 \dots x_n)$ $p_{X_1 \dots X_n}(x_1 \dots x_n) = p_{X_1}(x_1) \cdot p_{X_2}(x_2) \dots p_{X_n}(x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$

then $X_1 X_2 \dots X_n$ are independent

If $X_1 X_2 \dots X_n$ are independent

$$E(X_1 \cdot X_2 \cdots X_n) = E(X_1) \cdot E(X_2) \cdots E(X_n)$$

Note also that the probability generating function of $S_n = X_1 + X_2 \dots + X_n$ when $X_1 X_2 \dots X_n$ are independent is

$$G_{S_n}(s) = E\left(s^{(X_1 + X_2 + \dots X_n)}\right) = E\left(s^{X_1} s^{X_2} \dots s^{X_n}\right) =$$

$$= E(s^{X_1}) \cdot E(s^{X_2}) \dots E(s^{X_n}) =$$

$$= G_{X_1}(s) \cdot G_{X_2}(s) \dots G_{X_n}(s)$$

The sum of n independent Bernoulli(p) random variable is a Binomial(n, p),

Suppose that $X_i \sim$ Bernoulli(p) for $i = 1 - n$.

Moreover suppose that $X_1 \dots X_n$ are independent

Let $S_n = \sum_{i=1}^{n} X_i$. The p.g.f of X_i is $G_{X_i}(s) = (1-p+ps)$

The p.g.f of S_n is $G_{S_n}(s) = (1-p+ps) \cdot (1-p+ps) \cdots (1-p+ps)$

$= (1-p+ps)^n$ which is the pgf of a Binomial(n, p)

Theorem: Random sum formula Let N and $X_1, X_2 \ldots$ be independent random variables each taking values in $\{0, 1, \ldots\}$. If the X_i are identically distributed with common probability generating function G_X, then the sum

$$S = X_1 + X_2 + \cdots + X_N$$

has probability generating function

$$G_S(s) = G_N(G_X(s))$$

and

$$E(S) = E(N)E(X)$$

where $E(X)$ is the mean of a typical X_i

Proof

$$G_S(s) = E\left(s^{X_1+X_2+\dots X_N}\right) =$$

$$= \sum_{n=0}^{\infty} E\left(s^{X_1+\dots X_N} \mid N=n\right) \cdot P(N=n)$$

$$= \sum_{n=0}^{\infty} E\left(s^{X_1+X_2+\dots X_n}\right) P(N=n)$$

$$= \sum_{n=0}^{\infty} G_X(s)^n \, P(N=n) = G_N(G_X(s))$$

$E(X) =$

$= \sum E(X \mid B_i) \cdot P(B_i)$

$B_1, B_2, \dots$ was a partition

Note that $G_S'(s) = G_N'(G_X(s)) \, G_X'(s)$ $\quad s = 1$

and $G_S'(1) = G_N'(G_X(1)) \cdot G_X'(1) = G_N'(1) \cdot G_X'(1) = E(N) \cdot E(X)$