Stochastic Processes

Andrea Tancredi

Sapienza University of Rome

Week 6

Random variables

- A discrete random variable X on $(\Omega, \mathcal{F}, P)$ is a mapping $X\colon \Omega \to \mathbb{R}$ such that

  (a) $X(\Omega)$ is a countable set
  (b) $\{X = x\} = \{\omega \in \Omega : X(\omega) = x\} \in \mathcal{F}$

- A random variable X on $(\Omega, \mathcal{F}, P)$ is a mapping $X\colon \Omega \to \mathbb{R}$ such that

  $$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F} \quad \text{for all } x \in \mathbb{R}$$

  Note that

  $$\{\omega \in \Omega : X(\omega) \leq x\} = \{\omega \in \Omega : X(\omega) \in (-\infty, x]\} = \{X \in (\infty, x]\}$$

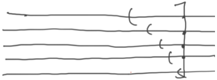
A discrete random variable is a random variable. In fact if X is discrete

$$\{\omega \in \Omega : X(\omega) \leq x\} = \{X \leq x\} = \bigcup_{y \in Im(X) : y \leq x} \{X = y\}$$

and the union belongs to $\mathcal{F}$ since is a countable union of events in $\mathcal{F}$

By the definition of random variable we can see that a lot of events have their own probabilities

- $\{X > a\} = \{X \leq a\}^c \in \mathcal{F}$ for all a since $\mathcal{F}$ is closed under complement
- If $a < b$, $\{X \in (a, b]\} = \{X > a\} \cap \{X \leq b\} \in \mathcal{F}$ for all a and b since $\mathcal{F}$ is closed under intersection
- Note also $x = \bigcap_n \left(x - \frac{1}{n}, x\right]$



hence

$$\{X = x\} = \left\{X \in \bigcap_n \left(x - \frac{1}{n}, x\right]\right\} = \bigcap_n \left\{X \in \left(x - \frac{1}{n}, x\right]\right\}$$

Distribution function

Definition If X is a random variable on $(\Omega, \mathcal{F}, P)$ the distribution function of X is the function $F_X : \mathbb{R} \to [0,1]$ defined by

$$F_X(x) = P(X \leq x)$$

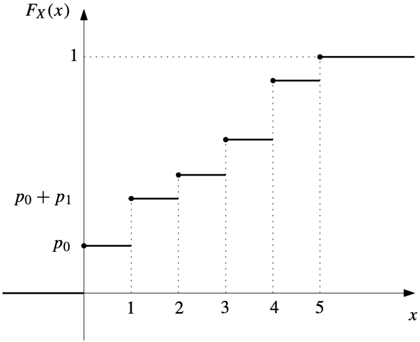
The distribution function is also referred to cumulative distribution function of X and in Italian *funzione di ripartizione di* X

Example Suppose that X is a discrete random variable taking non negative integer values and

$$P(X = k) = p_k \quad k = 0, 1,$$

Then $\{X \leq x\} \iff X \in \{0, 1, \ldots, \lfloor x \rfloor\}$ where $\lfloor x \rfloor$ is the integer part, i.e. the greatest integer not greater than x (if x=2.576, $\lfloor x \rfloor\} = 2$) and

$$F_X(x) = \begin{cases} 0 & x < 0 \\ p_1 + p_2 + \cdots + p_{\lfloor x \rfloor} & x \geq 0 \end{cases}$$



Properties of the distribution function

A distribution function $F_X(x)$ has the following properties

(i) $F_X(x)$ is monotonic non decreasing, $x \leq y \Rightarrow F_X(x) \leq F_X(y)$

(ii) $F_X(x) \rightarrow 0$ when $x \rightarrow -\infty$ and $F_x(x) \rightarrow 1$ when $x \rightarrow \infty$

(iii) $F_X(x)$ is continuous from the right

(i) To prove the first property note that if $x \leq y$

$$\{X \leq x\} \subseteq \{X \leq y\}$$

then since P is monotonic

$$F_X(x) = P(X \leq x) \leq P(X \leq y) = F_X(y)$$

(ii) For the second property note that when $x \to -\infty$ the event $X \leq x$ become less and less likely. In fact it converges to empty set hence by the the continuity theorem of the probability when $x \to -\infty$

$$F_X(x) = P(X \leq x) \to 0$$

Similarly, when $x \to \infty$ the event $X \leq x$ become more and more likely. In fact it converges to Ω hence by the the continuity theorem of the probability when $x \to \infty$

$$F_X(x) = P(X \leq x) \to 1$$

(iii) For the last property observe that

$$\{X \leq x\} = \bigcap_n \left\{X \leq x + \frac{1}{n}\right\}$$

and by the continuity theorem of probability

$$F_X(x) = P(X \leq x) = \lim_{n\to\infty} P(X \leq x + 1/n) = \lim_{n\to\infty} F_X(x + 1/n)$$

which is enough to guarantee right-continuity

- It is possible to prove that if F is a real function that satisfies properties (i), (ii) and (iii) there exists a probability space $(\Omega, \mathcal{F}, P)$ and a random variable X such that X has distribution F

Note that if $a < b$

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

In fact

$$(X \leq b) = (X \leq a) \bigcup (a < X \leq b)$$

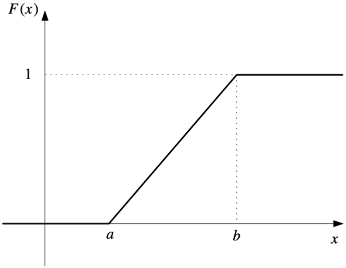
Then

$$\begin{aligned} F_X(b) &= P(X \leq b) = P(X \leq a) + P(a < X \leq b) \\ &= F_X(a) + P(a < X \leq b) \end{aligned}$$

Examples of distribution functions

- **Uniform distribution**

$$F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$



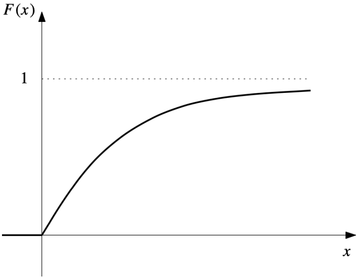
Note that if X is Uniform(a,b) and $0 \le x_1 \le x_2 \le 1$

$$P(x_1 < X \le x_2) = \frac{x_2 - a}{b-a} - \frac{x_1 - a}{b-a} = \frac{x_2 - x_1}{b-a}$$

Equal length intervals have the same probability

- **Exponential distribution**

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$$



If X is Exponential(λ), $P(X > t) = 1 - P(X \leq t) = 1 - F_X(t) = e^{-\lambda t}$
Then

$$\begin{aligned} P(X > x + t | X > x) &= \frac{P(X > x + t, X > x)}{P(X > x)} = \frac{P(X > x + t)}{e^{-\lambda x}} \\ &= \frac{e^{-\lambda(x+t)}}{e^{-\lambda x}} = e^{-\lambda t} \end{aligned}$$

Conditioning on $X > x$ the probability that X will last at least an extra amount of time greater than t is equal to the probability that X will last t from the origin

Continuous random variables

Definition A random variable X is continuous if its distribution function F_X can be written in the form

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u) du \quad for\, x \in \mathbb{R}$$

for some non negative functions f_X. The function f_X will be called the **density** of X

Example For the Exponential random variable we have

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases} \qquad f_X(x) = \begin{cases} 0 & x \leq 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$$

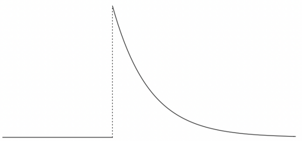
In fact, for $x \leq 0$

$$\int_{-\infty}^{x} f_X(u) du = \int_{-\infty}^{x} 0 du = c - c = 0 = F_X(x)$$

For $x > 0$

$$\int_{-\infty}^{x} f_X(u) du = \int_{0}^{x} \lambda e^{-\lambda u} du = -\lambda e^{-\lambda u} \Big|_{u=0}^{u=x} = 1 - e^{-\lambda x} = F_X(x)$$

$f_X(x)$

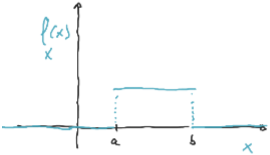


Example For the Uniform random variable we have

$$F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases} \qquad f_X(x) = \begin{cases} 0 & x \le a \\ \frac{1}{b-a} & a < x < b \\ 0 & x \ge b \end{cases}$$

In fact

$$F_X(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} \int_{-\infty}^{x} 0\,du = 0 & x \le a \\ \int_{-\infty}^{a} 0\,du + \int_{a}^{x} \dfrac{1}{b-a}\,du = \dfrac{x-a}{b-a} & a < x < b \\ \int_{-\infty}^{a} 0\,du + \int_{a}^{b} \dfrac{1}{b-a}\,du + \int_{b}^{x} 0\,du = 1 & x > b \end{cases}$$



In both the previous examples when $F'(x)$ exists we have $f_X(x) = F'(x)$. In this course we can take this as a rule

If X is a continuous random variable and F_X is *well behaved* we can say that the density is the derivative of the distribution function or more precisely

$$f_X(x) = \begin{cases} \dfrac{d}{dx} F_X(x) & \text{if this derivative exists at } x \\ 0 & \text{otherwise} \end{cases}$$

Another important property of f_X is that

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

In fact

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \lim_{c \to \infty} \int_{-\infty}^{c} f_X(x)\, dx = \lim_{c \to \infty} F_X(c) = 1$$

Exercise Let X be a continuous random variable with density function $f_X(x) = kx^2$ for $x \in (0,1)$ and 0 otherwise.

1. Find the value of k
2. Find the distribution function

To find k we know that

$$1 = \int_{-\infty}^{\infty} f_X(u)\,du = \int_0^1 k u^2\,du = k\int_0^1 u^2\,du = k \left.\frac{u^3}{3}\right|_0^1 =$$

$$= k\left[\frac{1}{3} - 0\right] = \frac{k}{3} \qquad k = 3$$

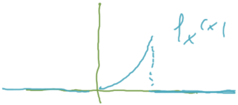
$$f_X(x) = \begin{cases} 3x^2 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \frac{k}{3}$$

$$3 = \cancel{3}\cdot\frac{k}{\cancel{3}} = k$$

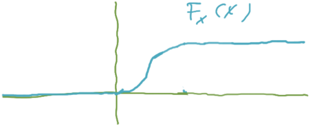
(2)

$$f_X(x) = \begin{cases} 0 & x \leq 0 \\ 3x^2 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$



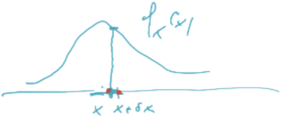
$$F_X(x) = \begin{cases} \int_{-\infty}^{x} 0 \, du = 0 & x < 0 \\ \int_{-\infty}^{0} 0 \, du + \int_{0}^{x} 3u^2 \, du = 0 + u^3 \Big|_0^x = x^3 & 0 \leq x \leq 1 \\ \int_{-\infty}^{0} 0 \, du + \int_{0}^{1} 3u^2 \, du + \int_{1}^{x} 0 \, du = 0 + 1 + 0 = 1 & \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ x^3 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$



other properties

$$P(x < X \leq x + \delta x) \approx f_X(x) \cdot \delta x$$

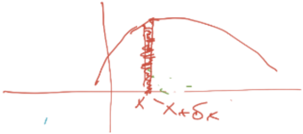


Infact

$$P(x < X \leq x + \delta x) = F_X(x + \delta x) - F_X(x) =$$

$$= \int_{-\infty}^{x+\delta x} f_X(u)\,du - \int_{-\infty}^{x} f_X(u)\,du = \int_{x}^{x+\delta x} f_X(u)\,du$$

(when $\delta x \to 0$) $= f_X(x) \cdot \delta x$



Theorem If X is a continuous random variable with density function f_X

$$P(X = x) = 0 \quad for\, x \in \mathbb{R}$$

$$P(a \leq X \leq b) = \int_a^b f_X(u)\, du \quad for\, a, b \in \mathbb{R}\ with\ a \leq b$$

Proof

$$\begin{aligned} P(X = x) &= P\left(\bigcap_{\epsilon > 0} \{x - \epsilon < X \leq x\}\right) = P(\lim_{\epsilon \to 0} \{x - \epsilon < X \leq x\}) \\ &= \lim_{\epsilon \to 0} P(\{x - \epsilon < X \leq x\}) = \lim_{\epsilon \to 0} [F_X(x) - F_X(x - \epsilon)] \\ &= \lim_{\epsilon \to 0} \int_{x-\epsilon}^{x} f_X(u) du = 0 \end{aligned}$$

Since $P(X = a) = 0$

$$\begin{aligned} P(a \leq X \leq b) &= P(a < X \leq b) \\ &= F_X(b) - F_X(a) = \int_a^b f_X(u)\, du \end{aligned}$$

- All random variables have a distribution function
- Discrete random variable have a probability mass function
- Continuous random variable have a density function
- but there are also random variables which are neither discrete nor continuous

Some common density functions

Book pag 68-69

- Uniform
- Exponential
- Normal
- Gamma
- Beta
- Cauchy
- Student-t
- ....

Note that for some of them is not trivial to show that the integral of the density on $\mathbb{R}$ is exactly 1. Other times the densities have a numerical constant which cannot be obtained analytically by which the integral is 1.