Stochastic Processes

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Week 7

Functions of continuous random variables

Let X be a random variable on $(\Omega, \mathcal{F}.P)$ and suppose that $g : \mathbb{R} \to \mathbb{R}$
Let $Y = g(X)$ be the mapping from $\Omega \to \mathbb{R}$ defined by

$$Y(\omega) = g\left(X(\omega)\right)$$

If g is *sufficiently well behaved* (such a continuous or a monotone function), then Y is a random variable

- If we know the distribution (or the density) of X it is possible to know the distribution (or the density) of Y?
- In the discrete case we have seen that

$$P(Y = y) = \sum_{x:g(x)=y} P(X = x) = \sum_{x \in g^{-1}(y)} P(X = x)$$

- and in the continuous case?

Example If X is continuous random variable with density function $f_X(x)$ and $g(x) = ax + b$ when $a > 0$ then $Y = aX + b$ has distribution function given by

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(aX + b \leq y) = P\left(X \leq \frac{y-b}{a}\right) \\ &= F_X(a^{-1}(y-b)) \end{aligned}$$

By differentiation with respect to y

$$f_Y(y) = f_X(a^{-1}(y-b))a^{-1} \quad \textit{for } y \in \mathbb{R}$$

For instance if $X \sim N(0,1)$, that is $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ for $x \in \mathbb{R}$ the density of $Y = aX + b$ is

$$f_Y(y) = a^{-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(a^{-1}(y-b))^2} = \frac{1}{\sqrt{2\pi a^2}} \exp\left(-\frac{1}{2a^2}(y-b)^2\right) \quad y \in \mathbb{R}$$

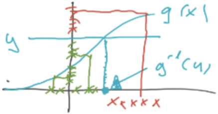
Theorem If X is a continuous random variable with density function f_X and g is a strictly increasing and differentiable function from $\mathbb{R}$ to $\mathbb{R}$, then $Y = g(X)$ has density function

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \frac{d}{dy}[g^{-1}(y)]$$

where $g^{-1}(y)$ is the inverse function of g

In fact

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P\left(X \le g^{-1}(y)\right) = F_X(g^{-1}(y))$$



and $f_Y(y) = F'_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$

Example Suppose that $X \sim N(0,1)$. Find the density of $Y = e^X$

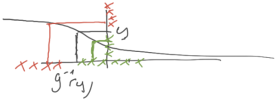
- The function $g(x)$ is e^x
- The inverse of $y = e^x$ is $g^{-1}(y) = \log y$
- The derivative of the inverse is $\frac{d}{dy} g^{-1}(y) = \frac{1}{y}$.
- The density of X is $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ for $x \in \mathbb{R}$,
- Then

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \frac{d}{dy}[g^{-1}(y)] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\log y)^2} \frac{1}{y} \quad y \in \mathbb{R}$$

If $g(x)$ is strictly decreasing

$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X > g^{-1}(y)) = 1 - P(X \leq g^{-1}(y)) =$

$= 1 - F_X(g^{-1}(y))$



$$f_Y(y) = \frac{d}{dy}(1 - F_X(g^{-1}(y))) = -f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Example Suppose that $X \sim Unif(0,1)$. Find the density of $Y = -log(X)$

There is not a formula for the density of $Y = g(X)$ with general classes of functions g...but specific cases can be easily handled

Example If X has density $f_X(x)$ and $g(x) = x^2$, then $Y = X^2$ has distribution function

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) \\ &= \begin{cases} 0 & y < 0 \\ P(-\sqrt{y} \leq X \leq \sqrt{y}) & y \geq 0 \end{cases} \\ &= \begin{cases} 0 & y < 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y \geq 0 \end{cases} \end{aligned}$$

The density $f_Y(y) = 0$ if $y \leq 0$ while for $y > 0$

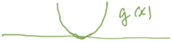
$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_y(y) \\ &= \frac{d}{dy}[F_X(\sqrt{y}) - F_X(-\sqrt{y})] = \frac{1}{2} f_X(\sqrt{y}) \frac{1}{\sqrt{y}} + \frac{1}{2} f_X(-\sqrt{y}) \frac{1}{\sqrt{y}} \\ &= \frac{1}{2} \frac{1}{\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})] \end{aligned}$$

The chi square χ^2 random variable

Exercise 5.55 (Book)

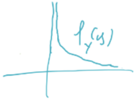
$X \sim \text{Normal}(0,1) \qquad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$Y = X^2 = g(X)$ where $g(x) = x^2$



$$f_Y(y) = \frac{1}{2} \cdot \frac{1}{\sqrt{y}} \cdot \left[f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right] =$$

$$= \frac{1}{2} \frac{1}{\sqrt{y}} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{y})^2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-\sqrt{y})^2} \right]$$



$$= \frac{1}{\cancel{2}} \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \cdot \cancel{2} = \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y}$$

Expectation of continuous random variables

Definition If X is a continuous random variable with density f_X the expectation of X is

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x) dx$$

whenever the integral converges absolutely, in that $\int_{-\infty}^{\infty} |x f_X(x)| dx < \infty$

Example If $X \sim Uniform(0, 1)$

$$f_X(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & otherwise \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x) dx = \int_{0}^{1} x\, 1\, dx = \frac{1}{2}\, x^2\Big|_0^1 = \frac{1}{2}$$

Integral by parts

$(fg)' = f'g + g'f \Rightarrow fg = \int f'g + \int g'f \quad \int f'g = fg - \int g'f$

Example (Exponential (λ))

Suppose that

$$f_X(x) = \begin{cases} 0 & x \leq 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} x \lambda e^{-\lambda x}\,dx = \Big[x \cdot (-e^{-\lambda x})\Big]_0^{\infty} - \int_0^{\infty} (-e^{-\lambda x})\,dx$$

$$= [0 - 0] + \int_0^{\infty} e^{-\lambda x}\,dx = -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty} = 0 + \frac{1}{\lambda} = \frac{1}{\lambda}$$

Theorem If X is a continuous random variable with density function $f_X(x)$ and $g : \mathbb{R} \to \mathbb{R}$ then

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

whenever the integral converges absolutely *no proof*

- If X is a continuous random variable then the expectation of $Y = aX + b$ is

$$E(Y) = \int_{-\infty}^{\infty} (ax + b) f_X(x)\, dx = aE(X) + b$$

- The variance of X is given by

$$Var(X) = E((X - E(X))^2) = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx$$

- The variance of $Y = aX + b$ is $Var(Y) = a^2 Var(X)$

Other important things to remember

- Also for continuous random variable $Var(X) = E(X^2) - E(X)^2$
- If $X \sim N(\mu, \sigma^2)$ $E(X) = \mu$ and $Var(X) = \sigma^2$
- Some random variables do not have a mean value, see the Cauchy distribution whose density is

$$f_X(x) = \frac{1}{\pi(1+x^2)} \quad x \in \mathbb{R}$$

Random vectors and independence

Given a probability space $(\Omega, \mathcal{F}, P)$ we can define a bivariate random variable (X, Y) which is a mapping from $\Omega \rightarrow R^2$

Definition The joint distribution function of (X, Y) is the mapping $F_{XY}(x, y) : \mathbb{R}^2 \rightarrow [0, 1]$ given by

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

Properties of the joint distribution function

$$\lim_{x,y \to -\infty} F_{X,Y}(x,y) = \lim_{\substack{x \to -\infty \\ y \to -\infty}} P(X \le x, Y \le y) = 0$$

$$\lim_{\substack{x \to \infty \\ y \to \infty}} F_{X,Y}(x,y) = \lim_{\substack{x \to \infty \\ y \to \infty}} P(X \le x, Y \le y) = P(\underbrace{X \le \infty, Y \le \infty}_{\text{the event } \Omega}) = 1$$

$F_{X,Y}(x,y)$ is a non-increasing function in each variable

$$F_{X,Y}(x_1, y_1) \le F_{X,Y}(x_2, y_2) \quad x_1 \le x_2 \quad y_1 \le y_2$$

From the joint distribution function we can obtain the distribution function $F_X(x)$ and $F_Y(y)$.

Infact

$$F_X(x) = P(X \leq x) = P(X \leq x, \underbrace{Y < \infty}_{\Omega}) = F_{X,Y}(x, \infty)$$

or better

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y)$$

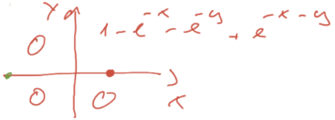
Similarly

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y)$$

$F_X(x)$ and $F_Y(y)$ in this context will be called marginal distributions

Exercise Suppose that $(X, Y)$is a bivariate random variable with joint distribution function

$$F_{XY}(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)} & x > 0\, y > 0 \\ 0 & otherwise \end{cases}$$



For $x \geq 0$

$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = \lim_{y\to\infty} 1 - e^{-x} - e^{-y} + e^{-x}e^{-y} = 1 - e^{-x}$$

For $x < 0$

$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = \lim_{y\to\infty} 0 = 0$$

Then $X \sim Exponential(\lambda)$

Definition We call X and Y independent if for all $x, y \in \mathbb{R}$ the event $\{X \leq x\}$ and $\{Y \leq y\}$ are independent

This means that X and Y are independent if *for* $x, y \in \mathbb{R}$

$$F_{XY}(x, y) = P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y) = F_X(x)F_Y(y)$$

Two random variables are independent if the joint distribution function is the product of the marginal distribution functions

We study families of random variables in very much the same way. Briefly, if $X_1, X_2, \dots, X_n$ are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, their *joint distribution function* is the function $F_{\mathbf{X}} : \mathbb{R}^n \to [0, 1]$ given by

$$F_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}\big(X_1 \le x_1,\ X_2 \le x_2, \dots,\ X_n \le x_n\big) \tag{6.9}$$

for $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. The variables $X_1, X_2, \dots, X_n$ are called *independent* if

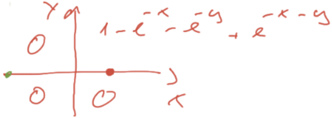
$$\mathbb{P}\big(X_1 \le x_1, \dots, X_n \le x_n\big) = \mathbb{P}(X_1 \le x_1) \cdots \mathbb{P}(X_n \le x_n) \qquad \text{for } \mathbf{x} \in \mathbb{R}^n,$$

or equivalently if

$$F_{\mathbf{X}}(\mathbf{x}) = F_{X_1}(x_1) \cdots F_{X_n}(x_n) \qquad \text{for } \mathbf{x} \in \mathbb{R}^n. \tag{6.10}$$

Exercise Consider (X, Y) with joint distribution function

$$F_{XY}(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)} & x > 0\; y > 0 \\ 0 & \textit{otherwise} \end{cases}$$



$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x} & x > 0 \end{cases} \quad \text{similarly} \quad F_Y(y) = \begin{cases} 0 & y \leq 0 \\ 1 - e^{-y} & y > 0 \end{cases}$$

$$F_X(x)F_y(y) = \begin{cases} 0 & x \leq 0, y > 0 \\ 0 & x > 0, y \leq 0 \\ 0 & x < 0, y < 0 \\ (1 - e^{-x})(1 - e^{-y}) = 1 - e^{-x} - e^{-y} + e^{-(x+y)} & x > 0, y > 0 \end{cases}$$

X and Y are independent

Joint density function

Definition The pair X, Y of random variables is called jointly continuous if its joint distribution function can be written as

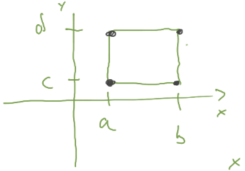
$$F_{XY}(x, y) = P(X \leq x, Y \leq y) = \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f(u, v) du\, dv$$

for $x, y \in \mathbb{R}$ and some function $f : \mathbb{R}^2 \to [0, \infty)$. If this holds, X, Y have joint density function given by $f(u, v)$ often denoted with $f_{XY}(u, v)$

Note that when the double integral is an integral of the kind $\int_a^b \int_c^d f(u,v)\, du\, dv$ we have that

$$\int_a^b \int_c^d f(u,v)\, du\, dv = \int_a^b \left[\int_c^d f(u,v)\, dv \right] du = \int_c^d \left[\int_a^b f(u,v)\, du \right] dv$$

The double integral is the volume under the surface $f(u,v)$ on the base rectangle (a,c) (b,c) (a,d) (b,d)



We can change the order of integration also when $a = -\infty$ and $d = -\infty$. Hence

$$\int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, du\, dv = \int_{-\infty}^{x} \left[\int_{-\infty}^{y} f(u,v)\, dv \right] du$$

$$= \int_{-\infty}^{y} \left[\int_{-\infty}^{x} f(u,v)\, du \right] dv$$

Other examples

$$\int_0^1 \int_0^1 x\,y \; dx\,dy = \frac{1}{4}$$

Note that

$$\frac{\partial}{\partial x}\int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv = \frac{\partial}{\partial x}\int_{-\infty}^{x}\left[\int_{-\infty}^{y} f(u,v)\,dv\right]du$$

$$= \int_{-\infty}^{y} f(x,v)\,dv$$

(that is the function $\int_{-\infty}^{y} f(u,v)\,dv$ evaluated when $u = x$)

then

$$\frac{\partial^2}{\partial y\,\partial x} F_{X,Y}(x,y) = \frac{\partial^2}{\partial y\,\partial x}\int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy = \frac{\partial}{\partial y}\int_{-\infty}^{y} f(x,v)\,dv = f(x,y)$$

Another exercise with double integrals

Find the value of the integral $\iint_A (x^2+\frac{1}{2}xy)\,dx\,dy$ where A:

$$\int_0^1 \int_0^2 (x^2+\tfrac{1}{2}xy)\,dy\,dx =$$

$$= \int_0^1 x^2y\Big|_0^2 + \frac{1}{2}x\,\frac{1}{2}y^2\Big|_0^2 \,dx =$$

$$= \int_0^1 \left(2x^2 + \frac{1}{\cancel{4}}x\cancel{4}\right) dx = \int_0^1 2x^2 + x \,dx = \frac{2}{3}x^3 + \frac{1}{2}x^2\Big|_0^1 =$$

$$= \frac{2}{3} + \frac{1}{2} = \frac{4+3}{6} = \frac{7}{6}$$

Consider a set

$$T = \{x, y : a \leq x \leq b,\, \alpha(x) \leq y \leq \beta(x)\}$$

then

$$\iint_{(x,y)\in T} f(x,y)\, dxdy = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} f(x,y) dy \right) dx$$

Consider a set

$$D = \{x, y : c \leq y \leq d,\, \gamma(y) \leq x \leq \delta(y)\}$$

then

$$\iint_{(x,y)\in D} f(x,y)\, dxdy = \int_c^d \left(\int_{\gamma(y)}^{\delta(y)} f(x,y) dx \right) dy$$

If $T = D$ we can use both the streets

Properties of the joint density function

Suppose that the random variables X and Y are jointly continuous. Then

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) = \int_{u=-\infty}^{x} \int_{y=-\infty}^{y} f_{XY}(u,v)\, du\, dv$$

Note that

- f_{XY} is given by the second order mixed partial derivatives of F_{XY}

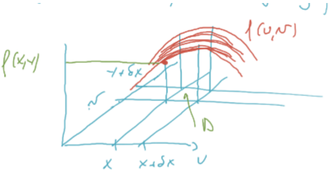
$$f_{XY}(x,y) = \begin{cases} \dfrac{\partial^2}{\partial y \partial x} F_{XY}(x,y) & \textit{if exists} \\ 0 & \textit{otherwise} \end{cases}$$

- f_{XY} is non negative $f_{XY}(x,y) \geq 0$ for $x\ y \in \mathbb{R}$
- The double integral of f_{XY} over $\mathbb{R}^2$ is 1

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(u,v)\, du\, dv = 1$$

- Similarly to the univariate case

$$P(x < X \leq x + \delta x, y < Y \leq y + \delta y) = f_{XY}(x, y)\delta x \delta y$$



$P((X,Y) \in D) = f(x,y)\delta x \delta y$

where $D = [x, x+\delta x] \times [y, y+\delta y]$

and $\delta x \to 0$ $\delta y \to 0$

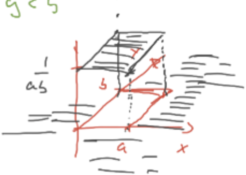
- More generally, if A is a regular subset of R^2 and X and Y are jointly contintinuous random variables

$$P((X, Y) \in A)) = \iint_{(x,y)\in A} f_{XY}(u, v) du dv$$

Bivariate Uniform random variable

Consider $a>0$, $b>0$. The function

$$f(x,y) = \begin{cases} \frac{1}{ab} & \text{if } 0<x<a \text{ and } 0<y<b \\ 0 & \text{otherwise} \end{cases}$$



is a joint density function.

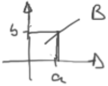
In fact $f(x,y) \geq 0$ $\forall x, y$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^a \left(\int_0^b \frac{1}{ab}\,dy\right) dx = \frac{1}{ab}\int_0^a y\Big|_0^b\,dx =$$

$$= \frac{1}{ab}\int_0^a b\,dx = \frac{1}{a}\int_0^a dx = \frac{1}{a}\,x\Big|_0^a = \frac{a}{a} = \underline{1}$$

In general we say that (X,Y) is uniform on the set B if

$$f(x,y) = \begin{cases} \frac{1}{\text{area}(B)} & \text{if } x,y \in B \\ 0 & \text{otherwise} \end{cases}$$

If $B = [0,a] \times [0,b]$ 

we have the density function seen in the previous slide

Note that if (X,Y) is uniform on B

$$P((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy = \iint_{A\cap B} f(x,y)\,dx\,dy =$$

$$\to = \frac{1}{\text{area of } B} \cdot \iint_{A\cap B} 1 \, dx\,dy = \frac{\text{area}(A\cap B)}{\text{area}(B)}$$

Suppose that B is the set given by x, y: $x^2+y^2 \le 1$

$$x^2+y^2=1$$

A is the set (x,y) : $x>0, y>0$ $x^2+y^2 \le 1$

If (X,Y) is uniform on B $P\left((X,Y) \in A\right) = \frac{1}{4} = \frac{\text{area}(A \cap B)}{\text{area of } B}$

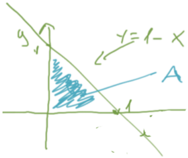
$$= \frac{\text{area}(A)}{\text{area}(B)}$$

Exercise 6.26 Suppose that (X, Y) have joint density function

$$f(x,y) = \begin{cases} e^{-x-y} & x > 0\ y > 0 \\ 0 & \textit{otherwise} \end{cases}$$

Find $P(X + Y \le 1)$ and $P(X > Y)$

$P(X+Y\le 1) = P\big((X,Y)\in A\big)$ where $A = \{(x,y): x+y\le 1\} = \{(x,y): y \le 1-x\}$



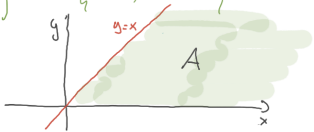
$$= \iint_A f(x,y)\,dx\,dy = \int_0^1 \left(\int_0^{1-x} e^{-x-y}\,dy\right)dx = \qquad -e^{-y}\Big|_{y=0}^{1}$$

$$= \int_0^1 e^{-x}\left(\int_0^{1-x} e^{-y}\,dy\right)dx = \int_0^1 e^{-x}\left(-e^{-y}\Big|_0^{1-x}\right)dx = \int_0^1 e^{-x}\left(1-e^{-(1-x)}\right) =$$

$$= \int_0^1 e^{-x} - e^{\cancel{-x}-1\cancel{+x}}\,dx = \int_0^1 \left(e^{-x}-e^{-1}\right)dx = -e^{-x}\Big|_0^1 - e^{-1}x\Big|_0^1 =$$

$$= 1 - e^{-1} - e^{-1} = 1 - 2e^{-1} = 1 - \frac{2}{e}$$

$P(X>Y) = P((X,Y)\in A)$ $\quad A=\{(x,y): x>y\}$



$$P(X>Y) = \int_0^\infty \left(\int_0^x e^{-x-y}\,dy\right)dx = \int_0^\infty e^{-x}\int_0^x e^{-y}\,dy\,dx = \int_0^\infty e^{-x}\left(-e^{-y}\Big|_0^x\right)dx$$

$$= \int_0^\infty e^{-x}(1-e^{-x})\,dx = \int_0^\infty e^{-x}dx - \int_0^\infty e^{-2x}\,dx = 1-\frac{1}{2}\cdot\int_0^\infty 2e^{-2x}dx$$

$$= 1-\frac{1}{2} = \frac{1}{2}$$

Marginal density functions and independence

if X, Y is a jointly continuous random variable

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

In fact

$$\begin{aligned}
f_X(x) &= \frac{d}{dx} F_X(x) = \frac{d}{dx} P(X \leq x) = \frac{d}{dx} P(X \leq x, Y \leq \infty) \\
&= \frac{d}{dx} \int_{u=-\infty}^{x} \int_{v=-\infty}^{\infty} f_{XY}(u,v)\,du\,dv \\
&= \frac{d}{dx} \int_{-\infty}^{x} \left(\int_{-\infty}^{\infty} f_{XY}(u,v)\,dv \right) du \\
&= \int_{-\infty}^{\infty} f_{XY}(x,v)\,dv
\end{aligned}$$

Note that if X and Y are independent then

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = P(X \le x) \cdot P(Y \le y) = F_X(x) \cdot F_Y(y)$$

and

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_X(x) \cdot F_Y(y) = f_X(x)\, f_Y(y) \quad \forall x\, y$$

Now suppose that $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ $\forall x, y$. Then

$$F_{X,Y}(x,y) = \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f_X(u)\, f_Y(v)\, du\, dv = \int_{u=\infty}^{x} f_X(u) \cdot F_Y(y)\, du = F_X(x) \cdot F_Y(y)$$

X and Y are independent if and only if $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ $\forall x, y$

Theorem. Jointly continuous random variables X and Y are independent if and only if their joint density function may be expressed in the form

$$f_{X,Y}(x,y) = g(x) \cdot h(y) \quad \text{for } x, y \in \mathbb{R}$$

as the product of a function of the first variable and a function of the second. (If you want adapt the proof done in the discrete case)

- **Example 6.32** On the whiteboard
- **Written exam June 2021. Exercise 3, points a,b and c**. On the whiteboard
- **Written exam June 2021. Exercise 4, points a, marginal of X**. On the whiteboard