Stochastic Processes

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Week 8

Sums of continuous random variables

Changes of variables

Conditional density function

Example 6.33 Suppose that X and Y have joint density function

$$f(x,y) = \begin{cases} ce^{-x-y} & 0 < x < y \\ 0 & \textit{otherwise} \end{cases}$$

Find the value of c and ascertain whether X and Y are independent

To do on the whiteboard

- **Exercise 6.35** Let X and Y have joint density function

$$f(x,y) = \begin{cases} cx & \textit{if } 0 < y < x < 1 \\ 0 & \textit{otherwise} \end{cases}$$

1. Find the value of the constant c.

$$1 = \int_0^1 \left(\int_0^x cx\, dy \right) dx = c \int_0^1 x^2\, dx = \frac{c}{3} \quad \Rightarrow c = 3$$

2. Find the marginal density of X and Y

$$f_X(x) = \int_0^x 3x\, dy = 3x^2 \quad x \in (0,1)$$

$$f_y(y) = \int_y^1 3x\, dx = \frac{3}{2}(1 - y^2) \quad y \in (0,1)$$

3. Are X and Y independent?
   No

Consider the integral $\int f(x)\,dx$. We can solve the integral by setting $x=\varphi(t)$ and calculating $\int f(\varphi(t))\,\varphi'(t)\,dt$ and substituting $t=\varphi^{-1}(x)$ in the solution (with respect to t)

For example consider

$$\int \frac{1}{e^{x}+e^{-x}}\,dx$$

we set $x=\log t$. Note that $t=e^{x}$

$$\int \frac{1}{t+\frac{1}{t}}\,\frac{1}{t}\,dt=\int \frac{1}{1+t^{2}}\,dt=\operatorname{arctg} t+c$$

Then $\int \frac{1}{e^{x}+e^{-x}}\,dx = \operatorname{arctg}(e^{x})+c$

Infact $\int f(\varphi(t))\,\varphi'(t)\,dt = F(\varphi(t))$ where $F(x) = \int f(x)\,dx$

then $F(\varphi(t))\Big|_{t=\varphi^{-1}(x)} = F(\varphi(\varphi^{-1}(x))) = F(x)$

$\square$

Definite integral

$$\int_a^b f(x)\,dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\varphi'(t)\,dt$$

$y = t - k \quad t = y + k$

$$\int_a^b f(y)\,dy = \int_{a+k}^{b+k} f(\underbrace{t-k}_{\varphi(t)}) \cdot \underbrace{1}_{\varphi'(t)}\,dt$$

$y = t - x \quad t = y + x$

$$\int_{-\infty}^{z-x} f(y)\,dy = \int_{-\infty}^{z} f(t-x)\,dt$$

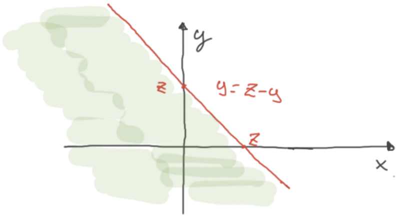
Sums of continuous random variables

Suppose that X, Y have joint density function $f_{XY}(x, y)$. Consider $Z = X + Y$.

$$\begin{aligned} P(Z \le z) &= P(X + Y \le z) = P(X + Y \in A) \\ &= \iint_A f_{XY}(x, y)\,dx\,dy \end{aligned}$$

where

$$\begin{aligned} A &= \left\{(x, y) \in \mathbb{R}^2 : x + y \le z\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : -\infty < x < \infty, -\infty < y < z - x\right\} \end{aligned}$$



$$\begin{aligned} P(Z \le z) &= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f(x,y)\,dy \right) dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z} f_{XY}(x, t-x) dt \right) dx \\ &= \int_{-\infty}^{z} \left(\int_{-\infty}^{\infty} f_{XY}(x, t-x)\ dx \right) dt \end{aligned}$$

Then

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} P(Z \le z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x) dx$$

Note also that if X andd Y are independent $f_{XY}(x,y) = f_X(x) f_Y(y)$ and

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx \quad \textit{convolution formula}$$

Example 6.40 Let X and Y be independent variables with density

$$f_X(x) = \begin{cases} \dfrac{\lambda^s}{\Gamma(s)} e^{-\lambda x} x^{s-1} & x > 0 \\ 0 & otherwise \end{cases} \qquad f_Y(y) = \begin{cases} \dfrac{\lambda^t}{\Gamma(t)} e^{-\lambda y} y^{t-1} & y > 0 \\ 0 & otherwise \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$$

When $z < 0$ the density of $Z = X + Y$ is 0. When $z > 0$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = \int_0^z f_X(x) f_Y(z-x)\, dx \\ &= \frac{\lambda^{s+t}}{\Gamma(s)\Gamma(t)} \int_0^z x^{s-1} e^{-\lambda x} (z-x)^{t-1} e^{-\lambda(z-x)}\, dx \\ &= \frac{e^{-\lambda z}\lambda^{s+t}}{\Gamma(s)\Gamma(t)} \int_0^z x^{s-1} (z-x)^{t-1}\, dx \quad set\ y = x/z \\ &= \frac{e^{-\lambda z}\lambda^{s+t}}{\Gamma(s)\Gamma(t)} \int_0^1 (yz)^{s-1} (z-yz)^{t-1}\, z\, dy \end{aligned}$$

That is

$$
\begin{aligned}
f_Z(z) &= \frac{\lambda^{s+t}}{\Gamma(s)\Gamma(t)} \int_0^1 y^{s-1}(1-y)^{t-1}\,dy\; e^{-\lambda z} z^{s+t-1} \\
&\propto e^{-\lambda z} z^{s+t-1}
\end{aligned}
$$

This means that $f_Z(z)$ is proportional to a Gamma density with parameter $s+t$ and λ which is

$$
\frac{\lambda^{s+t}}{\Gamma(s+t)} e^{-\lambda z} z^{s+t-1}
$$

Hence we conclude that Z is Gamma(s+t,λ)

Exercise 6.45 If X and Y have joint density function

$$f(x,y) = \begin{cases} \frac{1}{2}(x+y)e^{-x-y} & if x, y > 0 \\ 0 & otherwise \end{cases}$$

find the density of $X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx =$$

$$= \int_0^{\infty} f_{X,Y}(x, z-x)\,dx \quad \text{sinca } x > 0$$

$$= \int_0^{z} f_{X,Y}(x, z-x)\,dx \quad \text{sin } y > 0$$

$$= \int_0^{z} \frac{1}{2}(x + z - x)\, e^{-x-z+x}\,dx =$$

$$= \int_0^{z} \frac{1}{2} z e^{-z}\,dx = \frac{1}{2} z^2 e^{-z} \quad z > 0$$

$$\frac{1}{2} z^{3-1} e^{-z} \Rightarrow$$

$$Z \sim Gamma(3,1)$$

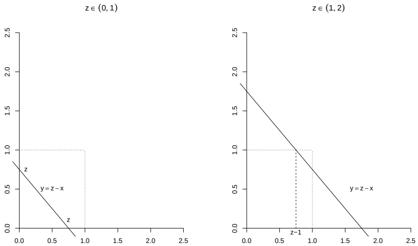
Example Let X and Y be independent Uniform(0,1) random variables
Find the density of $Z = X + Y$
The density of X and Y are

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & otherwise \end{cases} \qquad f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & otherwise \end{cases}$$

In calculating

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx$$

note that from the density of X x is such that $0 < x < 1$ and from the density of Y we have that $0 < z - x < 1$.



Then

$$f_Z(z) = \begin{cases} \int_0^z dx = z & 0 < z < 1 \\ \int_{z-1}^1 dx = 2 - z & 1 < z < 2 \end{cases}$$

Exercise Let X and Y be independent random variables where $X \sim \textit{Exponential}(\lambda)$ and $Y \sim \textit{Exponential}(\mu)$. Consider $Z = X + Y$

1. Show that

$$f_Z(z) = \begin{cases} \dfrac{\lambda\mu}{\lambda - \mu}(e^{-\mu z} - e^{-\lambda z}) & z > 0 \\ 0 & \textit{otherwise} \end{cases}$$

2. Find the density of Z when $\mu = \lambda$
3. Find the expected value of Z

Changes of variables

- Consider a jointly continuous random variable X, Y with density $f_{XY}(x, y)$
- Take two transformation of X and Y

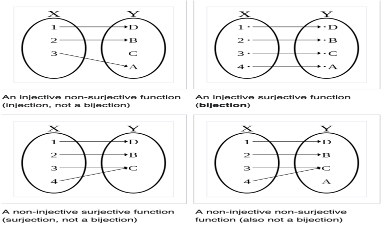
  $$U = u(X, Y) \quad V = v(X, Y)$$

  For example $U = X + Y$ and $V = X - Y$
- That is, suppose to apply to each realization x, y of X Y the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = (u(x, y), v(x, y))$
- Assume that the transformation T is a bijection between $D \subseteq \mathbb{R}^2$ and $S \subseteq \mathbb{R}^2$

A bijection T from D to S is a mapping such that

i) $\forall s \in S \; \exists w \in D : T(w) = s$ *(surjective)*

ii) if $T(w_1) = T(w_2)$ then $w_1 = w_2$ *(injective)*



An injective non-surjective function (injection, not a bijection)

An injective surjective function (**bijection**)

A non-injective surjective function (surjection, not a bijection)

A non-injective non-surjective function (also not a bijection)

A bijection $T : D \to S$ can be inverted, that is we have also a mapping $T^{-1} : S \to D$ where $T^{-1}(u, v) = (x, y)$ is exactly the point such that $T(x, y) = (u, v)$

Example of a bijection

Consider $(x,y) \in \mathbb{R}^2$, the mapping $T(x,y) = (x+y,\ x-y)$ is a bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$. To find the inverse mapping observe that

In fact $\begin{cases} u = x+y \\ v = x-y \end{cases}$ $\begin{cases} u+v = 2x \\ v = x-y \end{cases}$ $\begin{cases} x = \frac{1}{2}(u+v) \\ y = x-v \end{cases}$ $\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}u + \frac{1}{2}v - v \end{cases}$

then

$$x = \frac{1}{2}(u+v)$$

$$y = \frac{1}{2}(u-v)$$

Given the inverse mapping T^{-1} will need the Jacobian of T^{-1}, that is the determinant

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x(u,v)}{\partial u} & \dfrac{\partial x(u,v)}{\partial v} \\[2ex] \dfrac{\partial y(u,v)}{\partial u} & \dfrac{\partial y(u,v)}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial v}$$

Example $T(x,y) = (x+y, x-y)$ $\quad T^{-1}(u,v) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)$

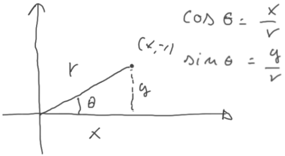
$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\[2ex] \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

By the theory of multiple integrals if $T(x, y) = (u(x, y),\ v(x, y))$ is bijective and $J(u, v)$ is the Jacobian of $T^{-1}(u, v) = (x(u, v),\ y(u, v))$ we have

$$\iint_A g(x, y)\, dx\, dy = \iint_{T(A)} g(x(u, v), y(u, v))|J(u, v)|\, du\, dv$$

Example For each point x, y consider the polar transformation θ, r where

$$x = r \cos\theta$$
$$y = r \sin\theta$$



The Jacobian is

$$J = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial \theta} = r\cos^2\theta + r\sin^2\theta = r$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}\, dx\, dy = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} r\, e^{-\frac{1}{2}r^2}\, dr\, d\theta = 1$$

Theorem: Jacobian formula Let X and Y be jointly continuous with joint density $f_{XY}(x, y)$, and let $D = \{(x, y) : f_{XY}(x, y) > 0\}$. If the mapping T given by

$$T(x, y) = (u(x, y), v(x, y))$$

is a bijection from D to the set $S \subseteq \mathbb{R}^2$, then the the pair

$$U = u(X, Y), V = v(X, Y)$$

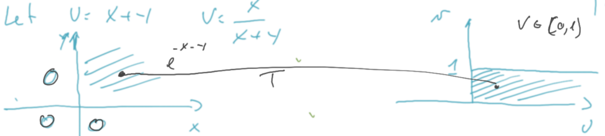
is jointly continuous with density function

$$f_{UV}(u, v) = \begin{cases} f_{XY}(x(u, v), y(u, v))|J(u, v)| & \textit{if } (u, v) \in S \\ 0 & \textit{otherwise} \end{cases}$$

Example 6.53 To do on the whiteboard

Example 6.53 Consider (X,Y) with density $f_{XY}(x,y) = \begin{cases} e^{-x-y} & x>0 \quad y>0 \\ 0 & \text{otherwise} \end{cases}$

Let $U = X+Y \qquad V = \dfrac{X}{X+Y}$



U is positive since it is $X+Y$, >0 >0

The mapping is given by $T(x,y) = (u,v)$ where $u = x+y$, $v = \dfrac{x}{x+y}$

Note that $v = \dfrac{x}{u}$, then $x = v\cdot u$. Moreover $y = u-x = u - v\cdot u = u(1-v)$

The Jacobian is
$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ (1-v) & -u \end{vmatrix} = -vu - u(1-v) = \\ = -vu - u + uv = -u$$

$$f_{UV}(u,v) = f_{X,Y}(x = vu, y = u(1-v)) \cdot |u| = e^{-vu-u+uv}\,|u| \qquad u>0, \; 0<v<1$$
$$= u\,e^{-u} \qquad u>0 \quad 0<v<1$$

$$f_{UV}(u,v) = \begin{cases} u e^{-u} & u>0 \quad 0<v<1 \\ 0 & \text{otherwise} \end{cases}$$

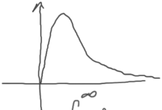
$U \sim \text{Gamma}(2,1)$

$V \sim \text{Uniform}(0,1)$

Let's find the marginal densities

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv = \int_0^1 u e^{-u}\,dv = u e^{-u} \int_0^1 dv = u e^{-u}$$

Gamma (2,1)



$$f_V(v) = \int_0^{\infty} f_{UV}(u,v)\,du = \int_0^{\infty} u e^{-u}\,du = \text{Mean of an exponential } (\lambda=1) \text{ density} = 1$$

$V \sim \text{Uniform}(0,1)$

Exercise 6.55 To do on the whiteboard
Let X, Y be a jointly continuous random variabe with density

$$f_{XY}(x,y) = \begin{cases} \frac{1}{4}e^{-\frac{1}{2}(x+y)} & x > 0\, y > 0 \\ 0 & otherwise \end{cases}$$

1. Find the joint density of U, V where

$$U = \frac{1}{2}(X - Y) \quad V = Y$$

2. Find the marginal density of U

- Consider $T(x, y) = (u(x, y), v(x, y))$ where $u(x, y) = \frac{1}{2}(x - y)$ and $v(x, y) = y$.
- The inverse transformation is $T^{-1}(u, v) = (x(u, v), y(u, v))$ where $x(u, v) = 2u + v$ and $y(u, v) = v$
- $D = \{(x, y) : x > 0, y > 0\}$. By applying T to the set D we have

$$S = T(D) = \{(u, v) : -\infty < u < \infty, v > 0, 2u + v > 0\}$$

- $J(u, v) = 2$
- The density of U, V is

$$f_{UV}(u, v) = \begin{cases} \frac{1}{4} e^{-\frac{1}{2}(2u+v+v)} \, 2 & (u, v) \in S \\ 0 & \textit{otherwise} \end{cases}$$

that is

$$f_{UV}(u, v) = \begin{cases} \frac{1}{2} e^{-(u+v)} & -\infty < u < \infty, \; v > 0, \; v > -2u \\ 0 & \textit{otherwise} \end{cases}$$

Let's find the marginal of U

If $u > 0$

$$f_U(u) = \int_0^\infty \frac{1}{2} e^{-u+v} dv = \frac{1}{2} e^{-u}$$

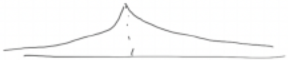
If $u < 0$

$$f_U(u) = \int_{-2u}^\infty \frac{1}{2} e^{-u+v} dv = \frac{1}{2} e^{u}$$

Then

$$f_U(u) = \frac{1}{2} e^{-|u|} \qquad -\infty < u < \infty$$

Then $f_U(u) = \frac{1}{2} e^{-|u|}$ $\quad -\infty < u < \infty$



$U \sim$ Double exponential

Conditional density function

With continuous variable we cannot calculate

$$P(Y \leq y|X = x)$$

by the formula

$$P(A|B) = P(A \cap B)/P(B)$$

since $P(B) = 0$

However we have that

$$\lim_{t \to 0} P(Y \leq y|x \leq X \leq x + t) = \int_{-\infty}^{y} \frac{f_{XY}(x.v)}{f_X(x)} \, dv$$

$$\lim_{t \to 0} P(Y \le y \mid x \le X \le x+t) = \lim_{t \to 0} \frac{P(Y \le y,\ x \le X \le x+t)}{P(x \le X \le x+t)} =$$

$$= \lim_{t \to 0} \frac{\int_x^{x+t} \left(\int_{-\infty}^{y} f_{X,Y}(u,v)\, dv \right) du}{\int_x^{x+t} f_X(u)\, du} \underset{\text{de l'Hopital}}{\overset{0/0}{=}} \lim_{t \to 0} \frac{\frac{d}{dt} \int_x^{x+t} \left(\int_{-\infty}^{y} f_{X,Y}(u,v)\, dv \right) du}{\frac{d}{dt} \int_x^{x+t} f_X(u)\, du}$$

$$= \frac{\lim_{t \to 0} \int_{-\infty}^{y} f_{X,Y}(x+t, v)\, dv}{\lim_{t \to 0} f_X(x+t)} = \frac{\int_{-\infty}^{y} f_{XY}(x,v)\, dv}{f_X(x)} = \int_{-\infty}^{y} \frac{f_{XY}(x,v)}{f_X(x)}\, dv$$

The function $G(y) = \int_{-\infty}^{y} \frac{f_{XY}(x.v)}{f_X(x)}\, dv$ is a distribution function and $f_{XY}(x,y)/f_X(x)$ is a density

Definition The conditional density of Y given $X = x$ is denoted by $f_{Y|X}(y|x)$ and defined by

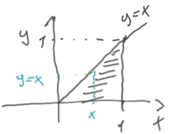
$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} \quad y \in \mathbb{R}$$

- If $f_X(x) > 0$, $f_{XY}(x,y) = f_X(x) f_{Y|X}(y|x)$
- if $f_Y(y) > 0$, $f_{XY}(x,y) = f_Y(y) f_{X|Y}(x|y)$
- If X and Y are independent, that is $f_{XY}(x,y) = f_X(x) f_Y(y)$

$$f_{Y|X}(y|x) = \frac{f_X(x) f_Y(y)}{f_X(x)} = f_Y(y) \quad f_{X|Y}(x|y) = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x)$$

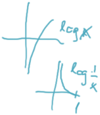
Example: Suppose that (X,Y) has joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{x} & 0<y<x<1 \\ 0 & \text{otherwise} \end{cases}$$



1) Verify that $\iint f_{X,Y}(x,y)\,dx\,dy = 1$

2) Find the marginal density of X and Y



3) Find the conditional density $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$

$$\int_0^1 \int_0^x \frac{1}{x}\,dy\,dx = \int_0^1 \frac{1}{x} \cdot y\Big|_0^x dx = \int_0^1 \frac{1}{x} \cdot x\,dx = 1$$

$$f_Y(y) = \int_y^1 \frac{1}{x}\,dx = \log x\Big|_y^1 = -\log y \qquad y \in (0,1)$$

$$f_X(x) = \int_0^x \frac{1}{x}\,dy = \frac{1}{x} \cdot x = 1 \qquad x \in (0,1)$$

$$f_{X|Y}(x|y) = \frac{1/x}{-\log y} = \frac{1}{x \log \frac{1}{y}} \qquad x \in (y,1)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1/x}{1} = \frac{1}{x}$$

$Y|X=x \sim$ Unif$(0,x)$

Unif$(a,b) \to f(x) = \frac{1}{b-a}$ Unif$(0,x)$ $f(y) = \frac{1}{x-0} = \frac{1}{x}$

- Example 6.59
- Exercise 6.60
- Exercise 6.61

Exercise 6.61

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-2x}\cdot 2e^{-2y} & x>0 \; y>0 \\ 0 & \text{otherwise} \end{cases}$$

X and Y independent and exponential with parameter λ

Find the joint density of $U = X$ $\quad V = X+Y$ and deduce that the conditional density of X given $X+Y=a$ is Uniform$(0,a)$

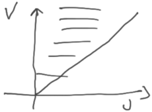
To find the joint density of U,V observe that

1) $V = X+Y = U+Y$ and Y is positive $\Rightarrow V>U$

2) the inverse transformation is $X=U$ $\quad Y = V-X = V-U$

$(u,v) = T(x,y) = (x, x+y)$ $\quad (x,y) = T^{-1}(u,v) = (u, v-u)$

3) The Jacobian is $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$



$$f_{X,Y}(x,y) = \lambda e^{-\lambda x} \, \lambda e^{-\lambda y} \qquad x>0 \quad y>0$$

$$f_{U,V}(u,v) = \lambda e^{-\lambda u} \, \lambda e^{-\lambda(v-u)} \, |1| \qquad v>u$$

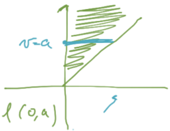
$$= \lambda^2 e^{-\lambda v} \qquad v>u$$

the marginal of U is $f_U(u) = \lambda e^{-\lambda u}$ ($U=X$ which is exponential)

we can check $f_U(u) = \int_u^\infty \lambda^2 e^{-\lambda v} \, dv = \lambda^2 \left. -e^{-\lambda v} \frac{1}{\lambda} \right|_{v=u}^{v=\infty} = \lambda e^{-\lambda u} \qquad u>0$

the marginal of V is

$$f_V(v) = \int_0^v \lambda^2 e^{-\lambda v} \, du = \lambda^2 e^{-\lambda v} \cdot v \qquad V \sim \text{Gamma}(2, \lambda)$$



$$f_{U|V}(u|v) = \frac{f_{UV}(u,v)}{f_V(v)} = \frac{\lambda^2 e^{-\lambda v}}{\lambda^2 e^{-\lambda v} \cdot v} = \frac{1}{v} \qquad u \in (0,v)$$

$U|V=a \sim \text{Unif}(0,a)$