Stochastic Processes

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Week 9

Expectations of continuous random variables

Bivariate normal distribution

Moments

Variance and covariance

Moment generating functions

Two inequalities

- Example 6.59. Find the conditional densities when the joint is

$$f(x, y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y < \infty \\ 0 & \textit{otherwise} \end{cases}$$

The marginal density of X is

$$f_X(x) = 2e^{-x} \int_x^\infty e^{-y}\, dy = 2e^{-2x} \quad x > 0$$

that is, $X \sim \mathit{Exp}(2)$. The marginal density of Y is

$$f_Y(y) = 2e^{-y} \int_0^y e^{-x}\, dx = 2e^{-y}(1 - e^{-y}) \quad y > 0$$

The conditional densities are

$$f_{Y|X}(x|y) = e^{-(y-x)} \quad y > x \quad \textit{and} \quad f_{X|Y}(x|y) = \frac{e^{-x}}{1 - e^{-y}} \quad 0 < x < y$$

- Exercise 6.60. Let X, Y be jointly continuous with density

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & otherwise \end{cases}$$

Find the conditional densities.

The marginal densities are

$$f_X(x) = \int_x^{\infty} e^{-y} dy = e^{-x} \quad x > 0$$

$$f_Y(y) = \int_0^{y} e^{-y} dx = ye^{-y} \quad y > 0$$

The conditional densities are

$$f_{Y|X}(x|y) = e^{-(y-x)} \quad y > x \quad and \quad f_{X|Y}(x|y) = \frac{1}{y} \quad 0 < x < y$$

- Exercise 6.61

Exercise 6.61

$f_{X,Y}(x,y) = \begin{cases} 2e^{-2x} \cdot 2e^{-2y} & x>0 \;\; y>0 \\ 0 & \text{otherwise} \end{cases}$

X and Y independent and exponential with parameter λ

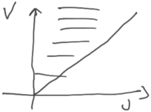
Find the joint density of $U = X$ $\quad V = X+Y$ and deduce that the conditional density of X given $X+Y=a$ is Uniform$(0,a)$

To find the joint density of U,V observe that

1) $V = X+Y = U+Y$ and Y is positive $\Rightarrow$ $V>U$

2) the inverse transformation is $X=U$ $\quad Y = V-X = V-U$

$(u,v) = T(x,y) = (x, x+y)$ $\quad (x,y) = T^{-1}(u,v) = (u, v-u)$



3) The Jacobian is $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

$f_{X,Y}(x,y) = \lambda e^{-\lambda x} \, \lambda e^{-\lambda y} \quad x>0 \quad y>0$

$f_{U,V}(u,v) = \lambda e^{-\lambda u} \, \lambda e^{-\lambda(v-u)} \, |1| \quad v>u$

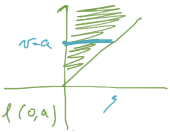
$= \lambda^2 e^{-\lambda v} \quad v>u$

the marginal of U is $f_U(u) = \lambda e^{-\lambda u}$ ($U=X$ which is exponential)

we can check $f_U(u) = \int_u^\infty \lambda^2 e^{-\lambda v} \, dv = \lambda^2 \left. -e^{-\lambda v} \frac{1}{\lambda} \right|_{v=u}^{v=\infty} = \lambda e^{-\lambda u} \quad u>0$

the marginal of V is

$f_V(v) = \int_0^v \lambda^2 e^{-\lambda v} \, du = \lambda^2 e^{-\lambda v} \cdot v \quad V \sim \text{Gamma}(2, \lambda)$



$f_{U|V}(u|v) = \frac{f_{UV}(u,v)}{f_V(v)} = \frac{\lambda^2 e^{-\lambda v}}{\lambda^2 e^{-\lambda v} \cdot v} = \frac{1}{v} \quad u \in (0,v)$

$U|V=a \sim \text{Unif}(0,a)$

Expectations of continuous random variables

Let X and Y be jointly continuous random variables on $(\Omega, \mathcal{F}, P)$, and let $g : \mathbb{R}^2 \to \mathbb{R}$. Consider the random variable $Z = g(X, Y)$.

Theorem We have that

$$E(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\, dx\, dy$$

whenever this integral converges absolutely. *no proof*

As consequence we have that also for a jointly continuous random variable X, Y

$$E(aX + bY) = aE(X) + bE(Y)$$

In fact..(on the whiteboard)

Note that also for a jointly continuous random variable X, Y, with X and Y independent we have

$$E(XY) = E(X)E(Y)$$

In fact..(on the white-board)

As for discrete case, the converse is not true

$$E(XY) = E(X)E(Y) \not\Rightarrow \textit{independence}$$

Theorem Jointly continuous random variables X and Y are independent if and only if

$$E(g(X)h(Y)) = E(g(X))E(h(Y))$$

for all functions $g, h : \mathbb{R} \to \mathbb{R}$ for which the expectation exists. (*no proof*)

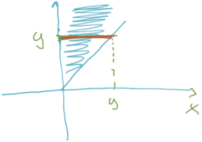
Definition The conditional expectation of Y given $X = x$, written $E(Y|X = x)$, is the mean of the conditional density function

$$E(Y|X = x) = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy = \int_{-\infty}^{\infty} y\, \frac{f_{XY}(x, y)}{f_X(x)}\, dy$$

for any value x for which $f_X(x) > 0$

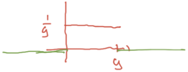
Exercise 6.72

$$f_{X,Y}(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$



$E(X|Y=y)$

the density of $X|Y=y$



Gamma(2,1)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

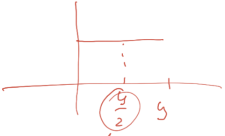
$$f_Y(y) = \int_0^y e^{-y}\, dx = e^{-y} \int_0^y 1\, dx = e^{-y}\, x \Big|_0^y = y e^{-y}$$

$$f_{X|Y}(x|y) = \frac{e^{-y}}{y e^{-y}} = \frac{1}{y} \quad x \in (0, y) \Rightarrow X|Y=y \sim \text{Unif}(0, y)$$

$$E(X \mid Y=y) = \int x \cdot f_{X|Y}(x \mid y)\, dx = \int_0^y x \cdot \frac{1}{y}\, dx = \frac{1}{y} \cdot \int_0^y x\, dx =$$

$\uparrow$ density of Uniform $(0, y)$

$$= \frac{1}{y}\, \frac{1}{2} x^2 \Big|_0^y = \frac{1}{\cancel{y}} \frac{y^{\cancel{2}}}{2} = \frac{y}{2}$$



mean of a Uniform $(0, y)$

Theorem If X and Y are jointly continuous random variables, then

$$E(Y) = \int E(Y|X = x) f_X(x)\, dx$$

where the integral is over all the values x such that $f_X(x) > 0$

$$E(Y) = \int y f_Y(y)\, dy = \int y \int f_{X,Y}(x,y)\, dx\, dy = \iint y\, f_X(x)\, f_{Y|X}(y|x)\, dx\, dy$$

$$= \int \left(\int y\, f_{Y|X}(y|x)\, dy \right) f_X(x)\, dx = \int g(x)\, f_X(x)\, dx$$

where $g(x) = E(Y|X=x)$ that is

$$E(Y) = \int E(Y|X=x)\, f_X(x)\, dx$$

considering the random variable $E(Y|X)$ we can say that

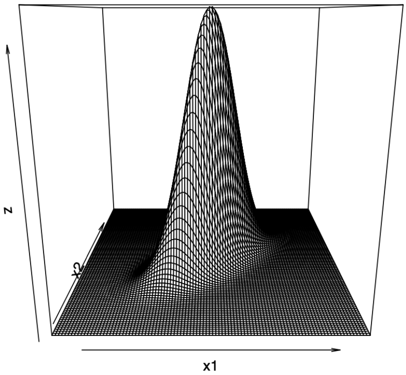
$$E(Y) = E(E(Y|X))$$

Bivariate normal distribution

The random variable X, Y with density

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right) \quad x, y \in \mathbb{R}$$

where $-1 < \rho < 1$ is called standard bivariate Normal



- Marginally, X and Y are standard Normal
- The conditional density of Y given $X = x$ is Normal with mean ρx and variance $1 - \rho^2$
- X and Y are independent if and only if $\rho = 0$

Exercise Let X and Y be independent random variables with densities

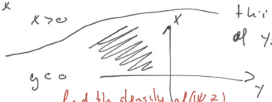
$$f_X(x) = \begin{cases} 4e^{-4x} & x > 0 \\ 0 & otherwise \end{cases} \qquad f_Y(y) = \begin{cases} e^{y} & y < 0 \\ 0 & otherwise \end{cases}$$

Find the density of W, Z where $W = 4X - Y$ and $Z = 4X + Y$

Exercise $X \sim$ Exponential(4) and Y independent

$f_X(x) = 4e^{-4x}$ $\quad x > 0$

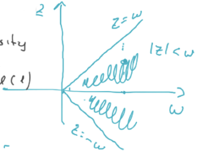
$f_Y(y) = e^{y}$ $\quad y < 0$



this is the density of $Y = -V$ where $V \sim$ Exponential(1)

find the density of (W, Z)

$$\begin{cases} W = 4X - Y & W + Z = 8X & X = \frac{1}{8}(W+Z) \\ Z = 4X + Y & Z - W = 2Y & Y = \frac{1}{2}(Z - W) \end{cases}$$



$-w < z < w$

Note the w is positive & z can vary between $-\infty, \infty$ and

$\{(w,z) : \frac{1}{8}(w+z) > 0, \ \frac{1}{2}(z-w) < 0 \quad w > 0 \ z \in \mathbb{R}\}$

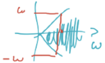
$= \{(w,z) : w + z > 0, \ z - w < 0, \ w > 0, z \in \mathbb{R}\} = \{(w,z) : w > -z, \ z < w, \ w > 0, z \in \mathbb{R}\}$

$= \{(w,z) : -w < z, \ z < w, \ w > 0, z \in \mathbb{R}\} = \{(w,z) : |z| < w, \ w > 0, z \in \mathbb{R}\}$

Jacobian $\begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{16} + \frac{1}{16} = \frac{2}{16} = \frac{1}{8}$

$$f_{W,Z}(w,z) = f_{X,Y}(x = x(w,z), y = y(w,z)) \cdot |J|$$

$$f_{W,Z}(w,z) = 4\, e^{-4\frac{1}{8}(w+z)}\, e^{\frac{1}{2}(z-w)}\, \frac{1}{8} \qquad |z| < w \qquad w > 0$$



$$= \frac{1}{2} e^{-\frac{1}{2}[w + z - z + w]} = \frac{1}{2} e^{-w} \qquad |z| < w \qquad w > 0$$

$$f_W(w) = \frac{1}{2} e^{-w} \int_{-w}^{w} 1\, dz = \frac{1}{2} e^{-w} z \Big|_{-w}^{w} = \frac{1}{2} e^{-w}\, 2w = w\, e^{-w}$$

$Z \sim$ double exponential

In fact $W = 4X - Y \sim \underbrace{Exp(1) + Exp(1)}_{\text{independent}} \sim Gamma(2,1)$

If $z > 0$

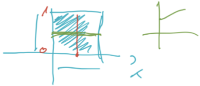
$$f_Z(z) = \int_z^\infty \frac{1}{2} e^{-w}\, dw = -\frac{1}{2} e^{-w} \Big|_z^\infty = \frac{1}{2} e^{-z}$$

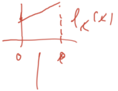
If $z < 0$

$$f_Z(z) = \int_{-z}^\infty \frac{1}{2} e^{-w}\, dw = \frac{1}{2} e^{z}$$

Ex 6 From exam october 2020

$$f_{XY}(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0<x<1 \quad 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$





$P(X>0.8)$ $\quad P(X>0.8 \mid Y=0.3)$ $\quad P(Y>0.8 \mid X=0.3)$

$$f_X(x) = \int_0^1 \frac{2}{5}(2x+3y)\,dy = \frac{2}{5}\left(2xy\Big|_0^1 + 3\frac{1}{2}y^2\Big|_0^1\right) = \frac{2}{5}\left(2x+\frac{3}{2}\right) = \frac{4}{5}x+\frac{3}{5}$$

$$f_Y(y) = \int_0^1 \frac{2}{5}(2x+3y)\,dx = \frac{2}{5}\left(2\frac{1}{2}x^2\Big|_0^1 + 3yx\Big|_0^1\right) = \frac{2}{5}(1+3y) = \frac{2}{5}+\frac{6}{5}y$$

$$P(X>0.8) = \int_{0.8}^1 f_X(x)\,dx = \frac{3}{5}x\Big|_{0.8}^1 + \frac{4}{5}\frac{1}{2}x^2\Big|_{0.8}^1 = \frac{3}{5}+\frac{4}{10}-\frac{3}{5}\frac{8}{10}-\frac{4}{10}\left(\frac{8}{10}\right)^2 = \frac{264}{1000} =$$

$$= 66/250 = 0.264$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{2}{5}(2x+3y)}{\frac{3}{5}+\frac{4}{5}x}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{2}{5}(2x+3y)}{2/5 + 6/5\,y}$$

$$P(X>0.8 \mid Y=0.3) = \int_{0.8}^{1} f_{X|Y}(x|y)\,dx = \int_{0.8}^{1} \frac{2}{5}\left(2x+\frac{9}{10}\right)dx \cdot \frac{1}{2/5+6/5\cdot\frac{3}{10}}$$

$$= \frac{1}{\frac{2}{5}+\frac{18}{50}} \cdot \frac{2}{5}\int_{\frac{8}{10}}^{1}\left(2x+\frac{9}{10}\right)dx = \frac{50}{20+18}\cdot\frac{2}{5}\cdot\left(2\,\frac{1}{2}x^2\Big|_{\frac{8}{10}}^{1} + \frac{9}{10}x\Big|_{\frac{8}{10}}^{1}\right)$$

$$= \frac{20}{38}\left(1-\frac{64}{100}+\frac{9}{10}-\frac{72}{100}\right) = \frac{20}{38}\,\frac{100-64+90-72}{100}$$

$$= \frac{20\cdot 54}{38\cdot 100} = \frac{54}{38\cdot 5} =$$

Exam exercise, January 2021 You choose a point (X, Y) where X is Uniform(0,1) and the density of $Y|X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} ky & \textit{if } \ y \in (0, x) \\ 0 & \textit{otherwise} \end{cases}$$

A. Find the marginal density of Y

B. Find the covariance of (X, Y)

C. Find the distribution of $Z = X + Y$ or alternatively that of $W = Y/X$

$$1 = \int_0^x ky\, dy = k \left. \frac{y^2}{2} \right|_1^x = \frac{k}{2} x^2$$

Hence $k = 2/x^2$,

$$f_{XY}(x, y) = \begin{cases} 2\, y/x^2 & 0 < y < x < 1 \\ 0 & \textit{otherwise} \end{cases}$$

and

$$f_Y(y) = \int_y^1 \frac{2\, y}{x^2}\, dx = -\, 2\, y \left. \frac{1}{x} \right|_y^1 = 2(1 - y) \quad y \in (0, 1)$$

$$E(Y|X=x) = \int y f_{Y|X}(y|x) dy = \int_0^x y \frac{2}{x^2} y \, dy = \frac{2x}{3}$$

$$E(Y) = E(E(Y|X)) = E(2X/3)) = \frac{2}{3} E(X) = \frac{2}{3}\frac{1}{2} = \frac{1}{3}$$

$$E(XY) = E(E(XY|X)) = E(X(2X/3)) = (2/3)E(X^2) = \frac{2}{3}\frac{1}{3} = \frac{2}{9}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{2}{9} - \frac{1}{2}\frac{1}{3} = \frac{1}{18}$$

Consider W. Since $Y < X$, the image of W is (0,1). For $w \in (0, 1)$

$$\begin{aligned} F_W(w) &= P(W \le w) = P(Y/X \le w) = P(Y \le wX) \\ &= \int_0^1 \left[\int_0^{wx} f_{XY}(x, y) dy \right] dx = \int_0^1 \frac{2}{x^2} \left[\int_0^{wx} y \, dy \right] dx \\ &= \int_0^1 \frac{2}{x^2} \frac{w^2 x^2}{2} dx = w^2 \end{aligned}$$

The density of W is then

$$f_W(w) = 2w \quad w \in (0, 1)$$

Moments

For any random variable X, the kth moment of X is the number $E(X^k)$, whenever this expectation exists

Example If X has the exponential distribution with parameter λ

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k \, \lambda \, e^{-\lambda x} dx (by \;\; parts) \\ &= \left[-x^k e^{-\lambda x}\right]_{x=0}^{x=\infty} + \int_0^\infty k x^{k-1} e^{-\lambda x} \, dx \\ &= 0 + \frac{k}{\lambda} \int_0^\infty x^{k-1} \lambda e^{-\lambda x} \, dx = \frac{k}{\lambda} E(X^{k-1}) \end{aligned}$$

and

$$E(X^0) = 1, E(X^1) = \frac{1}{\lambda}, E(X^2) = \frac{2}{\lambda}\frac{1}{\lambda}, E(X^3) = \frac{3}{\lambda}\frac{2}{\lambda}\frac{1}{\lambda}, \cdots$$

that is, the exponential distribution has moments of all orders, since

$$E(X^k) = \frac{k!}{\lambda^k}$$

There are also distributions that do not have moments

Example If X has the Cauchy distribution

$$E(X^k) = \int_{-\infty}^{\infty} \frac{x^k}{\pi(1+x^2)}\, dx$$

for values of k for which this integral converges absolutely.

Note that when $x \to \infty$ the integrand function is of the order of x^α with $\alpha = k - 2$ but

$$\int_1^\infty x^\alpha dx = \begin{cases} (\alpha+1)^{-1} x^{\alpha+1}\big|_{x=1}^{x=\infty} & \alpha \neq 1 \\ \log x\big|_{x=1}^{x=\infty} & \alpha = -1 \end{cases}$$

Hence the above integral is convergent only if $\alpha < -1$, that is with $\alpha = k - 2$ if $k < 1$

There are also distributions with only the first p moments

Example If X has density

$$f(x) = \frac{c}{1 + |x|^m} \quad x \in \mathbb{R}$$

where $m \geq 2$ and $c = (\int_{-\infty}^{\infty} \frac{dx}{1+|x|^m})^{-1}$ then X has only the moments of order k with $k < m - 1$, that is $\leq k \leq m - 2$

- Given the distribution function F_X of the random variable X, we may calculate its moments whenever they exist
- It is interesting to ask whether or not the converse is true: given the sequence $E(X), E(X^2)$,.... of (finite) moments of X , is it possible to reconstruct the distribution of X ?
- The general answer to this question is no, but is yes if we have some extra information about the moment sequence.

Theorem Suppose that all moments e $E(X), E(X^2)$,.... of the random variable X exist, and the the series

$$\sum_{k=0}^{\infty} \frac{1}{k!} t^k E(X^k)$$

is absolutely convergent for some $t > 0$. The the sequence of moments uniquely determines the distribution of X

Example Consider X with density

$$f(x) = \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}(\log x)^2} \quad \text{for } x > 0$$

that is X has the lognormal distribution

- X has finite moments of all orders...
- but the series $\sum_{k=0}^{\infty} \frac{1}{k!} t^k E(X^k)$ is not absolutely convergent
- In fact it is possible to find another density with the same moments of X

Variance and covariance

- The variance of X is $var(X) = E([X-\mu]^2)$ and it is a measure of dispersion about $E(X) = \mu$
- Note that $var(X) = 0$ if and only if $P(X = \mu) = 1$
- $var(X) = E(X^2) - \mu^2$
- $var(aX + b) = a^2 var(X)$
- $var(X + Y) = Var(X) + 2E(([X - E(X)][Y - E(Y)]) + Var(Y)$

- The covariance of X and Y is

$$cov(X, Y) = E(([X - E(X)][Y - E(Y)])$$

- $cov(X, Y) = E(XY) - E(X)E(Y)$
- Then $var(X + Y) = var(X) + var(Y) + 2cov(X, Y)$
- If X and Y are independent $cov(X, Y) = 0$ (but the converse is not true...we can have $E(XY) = E(X)E(Y)$ so that the covariance is 0 for dependent variables also)

- The correlation coefficient is of the random variables X and Y is the quantity $\rho(X, Y)$ given by

$$\rho(X, Y) = \frac{cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

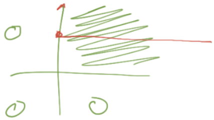
- The correlation coefficient is invariant for linear transformation

$$\rho(a + bX, c + dY) = \rho(X, Y)$$

- $-1 \leq \rho(X, Y) \leq 1$

Exercise (7.37) Consider (X,Y) with density

$$f(x,y)=\begin{cases}\frac{1}{y}e^{-y-x/y} & x>0 \quad y>0\\ 0 & \text{otherwise}\end{cases}$$



$\lambda e^{-\lambda x}$

Find $\mathrm{cov}(X,Y)$

$\mathrm{cov}(X,Y)=E(XY)-E(X)\cdot E(Y)$

$E(Y)=?$

density of $\mathrm{Exp}\left(\frac{1}{y}\right)$

$$f_Y(y)=\int_0^\infty \frac{1}{y}e^{-y}e^{-\frac{x}{y}}\,dx=e^{-y}\underbrace{\int_0^\infty \frac{1}{y}e^{-\frac{x}{y}}\,dx}_{\text{who is this?}}=e^{-y}\cdot 1=e^{-y}$$

$Y\sim \mathrm{Exp}(1)\quad E(Y)=\underline{1}$

To find $E(X)$ we first observe that

$$f_{X,Y}(x,y) = \underbrace{e^{-y}}_{f_Y(y)} \underbrace{\frac{1}{y} e^{-\frac{x}{y}}}_{f_{X|Y}(x,y)}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\cancel{e^{-y}} \frac{1}{y} e^{-\frac{x}{y}}}{\cancel{e^{-y}}} = \frac{1}{y} e^{-\frac{x}{y}} \leftarrow \text{density}$$

$\mathrm{Exp}(\lambda = \frac{1}{y})$

$E(\exp(\lambda)) = \frac{1}{\lambda}$

$$E(X | Y=y) = y \qquad E(X) = E(E(X|Y)) = E(Y) = \underline{1}$$

$$E(XY) = \iint xy \, f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty xy \cdot \frac{1}{y} e^{-y} e^{-\frac{x}{y}} \, dx\, dy = \int_0^\infty \left[\int_0^\infty xy \frac{1}{y} e^{-y} e^{-\frac{x}{y}} dx \right] dy$$

$$= \int_0^\infty e^{-y} y \underbrace{\int_0^\infty x \cdot \frac{1}{y} e^{-\frac{x}{y}} \, dx}_{} \, dy = \int_0^\infty e^{-y} \cdot y \cdot y \, dy =$$

$$= \int_0^\infty \underbrace{e^{-y} y^2 \, dy}_{E(Y^2)} = 2$$

do by yourself

$$\mathrm{cov}(X,Y) = E(XY) - E(X) \cdot E(Y) = 2 - 1 \cdot 1 = \underline{1}$$

Moment generating functions

Definition The moment generating function of the random variable X is the function M_X defined by

$$M_X(t) = E(e^{tX}),$$

for all $t \in \mathbb{R}$ for which this expectation exists

The moment generating function of X is then

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} P(X = x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

whenever this sum or integral converges absolutely. In some cases, the existence of $M_X(t)$ can pose a problem for non-zero values of t.

Example If X has the normal distribution with mean 0 and variance 1, then

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2-2tx)} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2-2tx+t^2)} dx \;\; e^{\frac{1}{2}t^2} \\
&= e^{\frac{1}{2}t^2} \quad \text{for all } t \in \mathbb{R}
\end{aligned}$$

Example If X has the Cauchy distribution

$$M_X(t) = \begin{cases} 1 & t = 0 \\ \infty & t \neq 0 \end{cases}$$

so that $M_X(t)$ exists only at $t = 0$

Other examples of moment generating functions

$X \sim \text{Exponential}(\lambda)$

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-x(\lambda - t)} dx =$$

$$= \lambda \frac{1}{\lambda - t} e^{-x(\lambda-t)} \Big|_\infty^0 = \begin{cases} \frac{\lambda}{\lambda - t} & t < \lambda \\ \infty & t \geq \lambda \end{cases}$$

$X \sim \text{Bernoulli}(p)$

$$M_X(t) = E(e^{tX}) = [e^{t0}(1-p) + e^{t1} p] = [(1-p) + p e^t] \quad \forall t$$

$X \sim \text{Binomial}(n, p)$

$$M_X(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k (1-p)^{n-k}$$

$$= [1-p+pe^t]^n \quad \text{by Binomial theorem}.$$

- It turns out to be important only that $M_X(t)$ exists in some neighbourhood $(-\delta, \delta)$ of the origin
- Hence we shall generally use moment generating functions subject to the assumption of existence in a neighbourhood of the origin.
- Observing that

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

we have (by rigorously interchanging the expectation and the series)

$$\begin{aligned} E(e^{tX}) &= E\left(1 + t\,X + \frac{1}{2}t^2X^2 + \frac{1}{3!}t^3X^3 + \cdots\right) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!}E(X^k) \end{aligned}$$

$M_X(t)$ is the exponential generating function of the moments of X

Theorem If $M_X(t)$ exists in some neighbourhood $(-\delta, \delta)$ of the origin, then for $k = 1, 2, \ldots$

$$E(X^k) = M_X^{(k)}(0) = \frac{d^k}{dt^k} M_X(t)|_{t=0}$$

the kth derivative of $M_X(t)$ evaluated at $t = 0$.

In fact, by observing that $\frac{d^k}{dt^k} e^{tx} = x^k e^{tx}$ and considering a continuous random variable

$$\begin{aligned} \frac{d^k}{dt^k} M_X(t) &= \frac{d^k}{dt^k} \int_{-\infty}^{\infty} e^{tx} f_x(x) dx \\ &= \int_{-\infty}^{\infty} \frac{d^k}{dt^k} e^{tx} f_X(x) dx \\ &= \int_{-\infty}^{\infty} x^k e^{tx} f_X(x) dx = E(X^k e^{tX}) \end{aligned}$$

If $t = 0$ we have $M_X^{(k)}(0) = E(X^k)$

Properties of the moment generating function

- If $Y = aX + b$

$$M_Y(t) = M_{aX+b}(t) = e^{tb} M_X(at)$$

- If X and Y are independent

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

- For the sum $S = X_1 + X_2 + \cdots + X_n$ of n independent

$$M_S(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t)$$

Theorem: Uniqueness theorem for moment generating function

If the moment generating function M_X satisfies

$$M_X(t) = E(e^{tX}) < \infty \quad -\delta < t < \delta$$

for some $\delta > 0$, there is a unique distribution with moment generating function M_X.

Furthemore, under this condition, we have $E(X^k) < \infty$ for $k = 1, 2, \dots$ and

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k) \quad -\delta < t < \delta$$

Theorem: Markov's inequality For any non negative random variable X

$$P(X \geq t) \leq \frac{E(X)}{t}$$

Consider the indicator random variable $I_A : \Omega \to \mathbb{R}$ where

$$I_A = \begin{cases} 1 & \textit{if } X(\omega) \geq t \\ 0 & \textit{otherwise} \end{cases}$$

Note that

- if $X < t$ then $t \cdot I_A = 0$ and $X \geq t \cdot I_A$
- if $X \geq t$ then $t \cdot I_A = t$ and $X \geq t \cdot I_A$
- Hence $X \geq t \cdot I_A$
- $E(t \cdot I_A) = t \cdot E(I_A) = t \cdot P(X \geq t)$
- If the random $U - V$ is a non negative random variable (that is $U \geq V$) $E(U - V) = E(U) - E(V) \geq 0 \Rightarrow E(U) \geq E(V)$
- Then $E(X) \geq t \cdot P(X \geq t)$, that is $P(X \geq t) \leq \frac{E(X)}{t}$

Theorem: Jensesn's inequality Let X be arandom variable taking values on (a, b) and let $g : (a, b) \to \mathbb{R}$ be a convex function. Suppose that both $E(X)$ and $E(g(X))$ exist. Then

$$E(g(X)) \geq g(E(X)$$

no proof

Examples with convex functions g

- $g(x) = x^2$, $E(X^2) \geq E(X)^2$
- $g(x) = -\log(x)$, $E(-log(X)) \geq -\log(E(X))$, that is $E(\log X) \leq \log E(X)$