Stochastic Processes

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Week 1

Experiments with chances

- An experiment is a process whose outcome and consequences are unknown
  - Draw a card, toss a coin, roll a dice...
  - ...
- A probability space formalizes all the features of an experiment
- To define a probability space we need
  (i) The set of all possible outcomes
  (ii) A list of all the events representing the possible consequences
  (iii) An assessment of the likelihoods of all the events

Example: a coin is tossed 2 times.
To define the probability space we need

(i) The set $\{HH, HT, TH, TT\}$ of possible outcomes ($H =$ head, $T =$tail)

(ii) A list of events such that
- no heads
- at least one head
- the second is tail

(iii) To say, for example, that each outcome is equally likely

Some general notations

- $\mathcal{E} =$ is the experiment of tossing a coin 2 times
- $\Omega = \{HH, HT, TH, TT\}$ is **the sample space** i.e. the set with all the possible outcomes
- $\omega =$ is a single outcome **elementary event**, for instance $\omega = HT$

Outcomes and events

- Questions about outcomes can be rewritten in terms of **subsets** of Ω
- For example: Has the outcome at least one head? Which is equivalent "Does the outcome lie in the subset $\{HT, TH, HH\}$
- Questions of interest are **events**

- To define the probability space we need to list all the **events (subsets)** which are interesting to us
- This list will be a **collection** of subsets of Ω
- If the subset A is in the list it means that we are interested in the event *the outcome of* $\mathcal{E}$ *lies in* A
- All the events are subsets of Ω
- A subset C of Ω occurs whenever the outcome of $\mathcal{E}$ lies in C

Subsets relationships and events

- **Union**: $A \cup B =$ *either A or B occurs*
- **Intersection**: $A \cap B =$ *both A and B occurs*
- **Complement set**: $A^c = \bar{A} = \Omega \backslash A =$ *A does not occur*

Example

- $A = \{HH, HT\} =$ *the first coin is head*
- $B = \{HH, TH\} =$ *the second coin is head*
- $A \cup B = \{HH, HT, TH\}$= *"either the first is H or the second is head" = "at lest one head"*
- $A \cap B = \{HH\}$= *" both the first an the second is head"= "no tails"*
- $(A \cup B)^c = \{TT\}$ = *" both the first and the second is tail"= "no heads"*

- $(A \cup B)^c = A^c \cap B^c$
  In fact

$$\omega \in (A \cup B)^c \Leftrightarrow \omega \notin (A \cup B) \Leftrightarrow \begin{cases} \omega \notin A \\ \omega \notin B \end{cases} \Leftrightarrow \omega \in A^c \cap B^c$$

- $(A \cap B)^c = A^c \cup B^c$ In fact from the previous relationship

$$(A^c \cup B^c)^c = A \cap B \quad \text{i.e.} \quad A^c \cup B^c = (A \cap B)^c$$

- $\bigcup_{i=1}^{\infty} A_i$ mens that A_i *occurs for some i*
- $\bigcap_{i=1}^{\infty} A_i$ means that A_i *occurs for every i*

- We use $\mathcal{F} = \{A_i, i \in I\}$ to indicate **collections** of subsets of Ω which are interesting to us
- Each $A \in \mathcal{F}$ is an **event**
- In some case $\mathcal{F}$ can be the **power set**, which is the set of all subsets of Ω ... other times $\mathcal{F}$ will be smaller
- In general if we are interested in A then we should be interested also in A^c. Similarly, if we are interested in A, B, C,... then we should be interested also in the event *at least one of A, B, C..*

Definition: Event space. The collection $\mathcal{F}$ of subset of the sample space Ω is an event space if

- $\mathcal{F}$ is non empty
- if $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
- if $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} \in \mathcal{F}$

It is possible to prove that Ω, , finite unions, finite intersection and countable intersections belong to $\mathcal{F}$

A set S is called countable if it may be put in one–one correspondence with a subset of the natural numbers $\{1, 2, 3, \ldots\}$.

Definition: Probability measure. A mapping $P : \mathcal{F} \to R$ is called a probability measure on $(\Omega, \mathcal{F})$ if

(a) $P(A) \geq 0 \; \forall \, A \in \mathcal{F}$

(b) $P(\Omega) = 1$ and $P(\emptyset) = 0$

(c) If $A_1, A_2, \ldots$ are disjoint sets in $\mathcal{F}$ (in that $A_i \cap A_j = \emptyset$ whenever $i \neq j$) then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Condition (c) requires that the probability of the union of a countable collection of disjoint sets is the sum of the individual probabilities. That is P is countably additive.

The second part of (b) is superfluous. In fact consider the disjoint events $A_1 = \Omega$ and $A_i = \emptyset$ for $i = 2, 3, \ldots$. Note that $\Omega = \cup_{i=1}^{\infty} A_i$. By the first part of (b) and (c) we have

$$\begin{aligned} 1 &= P(\Omega) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = P(A_1) + \sum_{i=2}^{\infty} P(A_i) = P(\Omega) + \sum_{i=2}^{\infty} P(\emptyset) \\ &= 1 + \sum_{i=2}^{\infty} P(\emptyset) \end{aligned}$$

Hence $P(\emptyset) = 0$

Note that for a sequence of disjoint sets A_i for $i = 1, \ldots m$ we have (by considering $A_i = \emptyset$ for $i > m$)

$$P\left(\bigcup_{i=1}^{m} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{m} P(A_i)$$

Hence the probability measure P is also finitely additive.

Example 1.16 Let $\Omega = \{\omega_1, \omega_2, ..., \ldots \omega_N\}$ be a finite set of exactly N points. Let $\mathcal{F}$ be the power set of Ω. The function P defined as

$$P(A) = \frac{1}{N}|A| \quad \text{for } A \in \mathcal{F} \quad \text{where} \quad |A| = \sum_{i:\omega_i \in A} 1$$

is a probability measure. In fact

- $P(A) \geq 0 \, \forall A$
- $$P(\Omega) = \frac{1}{N} \sum_{i:\omega_i \in \Omega} 1 = \frac{1}{N} N = 1$$
- Note that if $A \cap B = \emptyset$ $|A \cup B| = |A| + |B|$....similarly for disjoint sets
$$|\bigcup_{i=1}^{\infty} A_i| = \sum_{i=1}^{\infty} |A_i|$$
Hence
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \frac{1}{N}|\bigcup_{i=1}^{\infty} A_i| = \frac{1}{N}\sum_{i=1}^{\infty} |A_i| = \sum_{i=1}^{\infty} P(A_i)$$

Note the when Ω has n elements the number of possible subsets is 2^n.

Consider $\Omega = \omega_1, \omega_2, \omega_3$ the power set of Ω is the collection

$$\{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_3\}\}$$

In example 1.16

$$P(A) = \sum_{i:\omega_i \in A} \frac{1}{N}$$

Consider again $\Omega = \{\omega_1, \omega_2, ..., \ldots \omega_N\}$ and take $p_1, \ldots, p_N$ such that $\sum_{i=1}^{N} p_i = 1$. Also the function

$$Q(A) = \sum_{i:\omega_i \in A} p_i$$

is a probability measure (Exercise 1.17)

Probability spaces

Definition. A probability space is a triple $(\Omega, \mathcal{F}, P)$ of objects such that

(a) Ω is a non empty set

(b) $\mathcal{F}$ is an event space of subsets of Ω

(c) P is a probability measure on $(\Omega, \mathcal{F})$

Some properties

- If A and B are elements of $\mathcal{F}$, then $A \backslash B = A \cap B^c \in \mathcal{F}$
  In fact
  $$(A \cap B^c)^c = A^c \cup B \in \mathcal{F} \Rightarrow A \cap B^c \in \mathcal{F}$$
- If $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$
- If $A \in \mathcal{F}$ then $P(A) + P(A^c) = 1$ i.e. $P(A^c) = 1 - P(A)$
  In fact $\Omega = A \cup A^c$, then
  $$1 = P(\Omega) = P(A \cap A^c) = P(A) + P(A^c)$$

- If $A, B \in \mathcal{F}$ then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

In fact

$$\begin{aligned} P(A) &= P(A \backslash B) + P(A \cap B) \\ P(B) &= P(B \backslash A) + P(A \cap B) \end{aligned}$$

Then

$$\begin{aligned} P(A) + P(B) &= P(A \backslash B) + P(A \cap B) + P(B \backslash A) + P(A \cap B) \\ &= P(A \cup B) + P(A \cap B) \end{aligned}$$

- If A, $B \in \mathcal{F}$ and $A \subseteq B$ then $P(A) \leq P(B)$ In fact $B = A \cup B \backslash A$ hence

$$P(B) = P(A) + P(B \backslash A) \geq P(A)$$

- $A, B, C \in \mathcal{F}$. Note that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$
\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B \cup C) - P((A \cap (B \cup C)) \\
&= P(A) + P(B) + P(C) - P(B \cap C) \\
&- P((A \cap B) \cup (A \cap C)) \\
&= P(A) + P(B) + P(C) - P(B \cap C) \\
&- P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)
\end{aligned}
$$

Discrete sample spaces

If Ω is a coutable set, $\Omega = \{\omega_1, \omega_2, \ldots\}$ we normally take $\mathcal{F}$ to be the power set of Ω

In fact suppose that for each $\omega \in \Omega$ we are interested in whether or not this given ω is the outcome of $\mathcal{E}$; then we require that each singleton set $\omega \in \mathcal{F}$...this leads to consider the power set as the event space

Moreover for each set $A \in \mathcal{F}$ (which cannot be uncountable) we have that

$$A = \bigcup_{\omega \in A} \{\omega\}$$

and the probability of A is given by the collection of probabilities $P(\omega)$, $\omega \in \Omega$.
In fact, by the third rule of probability

$$P(A) = \sum_{\omega \in A} P(\omega)$$

Equiprobable outcomes

If $\Omega = \{\omega_1, \ldots, \omega_N\}$ and $P(\omega_i) = P(\omega_j) \ \forall (i,j)$ then

$$1 = P(\Omega) = P\left(\bigcup_{i=1}^{n} \omega_j\right) = \sum_{i=1}^{N} P(\omega_i) = NP(\omega_1)$$

so that

$$\frac{1}{N} = P(\omega_1) = p(\omega_j) \quad \forall j$$

and

$$P(A) = P\left(\bigcup_{i:\omega_i \in A} \omega_i\right) = \sum_{i:\omega_i \in A} P(\omega_i) = \sum_{i:\omega_i \in A} \frac{1}{N} = \frac{|A|}{N}$$

Permutations and combinations

The number of permutations of n distinct objects is

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1$$

The number of ordered subsequences of length r from these n objects is

$${}_nP_r = n(n-1)\cdots(n-(r-1) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

If the ordering of these r objects is not important, then any of the $r!$ possible orderings gives rise to the same subset.

The number of combinations ${}_nC_r$ (ways to choose r objects from n without looking at the order) and the ordered subsequences are related by the formula

$${}_nC_r \cdot r! = {}_nP_r$$

That is

$${}_nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r}!$$

Exercises

Consider the experiment where a coin is tossed n times

1. Find the cardinality of Ω

   $|\Omega| = 2^n$

2. What is the number of outcomes of Ω in which we have exactly k heads (Ex 1.25)

   $\binom{n}{k}$

3. If all possible outcomes are equiprobable (the coin is fair) what is the probability of getting k heads? And that of getting at least k heads?

$$P(\text{k heads}) = \binom{n}{k}\frac{1}{2^n} = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k}$$

$$P(\text{at least k heads}) = \sum_{r=k}^{n}\binom{n}{r}\frac{1}{2^n} = \sum_{r=k}^{n}\binom{n}{r}\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{n-r}$$

We distribute r indistinguishable balls int n cells at random, multiple occupancy can be permitted,

1. How many arrangements there are?
   $|\Omega| = n^r$
2. How many arrangements there are in which the first cell contains exactly k balls
   $\binom{r}{k}(n-1)^{r-k}$
3. What is the probability that the first cell contains exactly k balls

$$\frac{1}{n^r}\binom{r}{k}(n-1)^{r-k} = \binom{r}{k}\left(\frac{n-1}{n}\right)^{r-k}\left(\frac{1}{n}\right)^{k} = \binom{r}{k}\left(1-\frac{1}{n}\right)^{r-k}\left(\frac{1}{n}\right)^{k}$$