Stochastic Processes

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Week 10

The law of averages

Chebyshev's inequality and the weak law

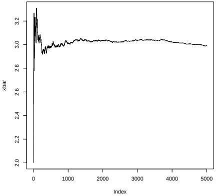
Central limit theorem

Random walks

The law of averages

- Consider the following R commands and look the resulting picture

```
> x=rpois(5000,3)
> xbar=cumsum(x)/(1:5000)
> plot(xbar,type="l")
>
```



- The averages of the results approach the underlying mean value
- Given a sequence $X_1, X_2, \ldots$ of independent and identically distributed random variables each having mean value μ

$$\frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$

when $n \to \infty$ *converges* to μ... but what does it mean *converges*

Definition We say that the sequence $Z_1, Z_2, \ldots$ of random variables converges in mean square to the (limit) random variable Z if

$$E([Z_n - Z]^2) \to 0 \quad \text{as } n \to \infty$$

If this holds we write $Z_n \to Z$ in mean square as $n \to \infty$

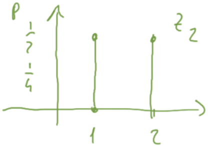
- Note that $E(Y^2) = 0$ if and only if $P(Y = 0) = 1$
  - $P(Y = 0) = 1 \Rightarrow E(Y^2) = 0$, obviously from the def. of E
  - $0 = E(Y^2) = \sum_y y^2 P(Y = y) \Rightarrow P(Y = 0) = 1$

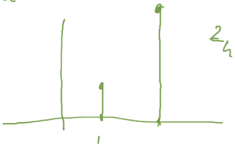
If $E([Z_n - Z]^2) \to 0$ then it follows that $Z_n - Z$ tends to 0 in some sense when $n \to \infty$

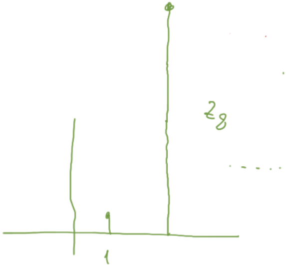
Example Consider the sequence of random variables

$Z_1, Z_2 \dots Z_n \dots$ where

$P(Z_n=1) = \frac{1}{n}$ $\qquad$ $P(Z_n=2) = 1-\frac{1}{n}$







Then consider the r.v. $Z=2$ (degenerate in 2) then

$$E\left([Z_n-Z]^2\right) = (1-2)^2 P(Z_n=1) + (2-2)^2 \cdot P(Z_n=2) =$$

$$= 1 \cdot \frac{1}{n} + 0 \cdot \left(1-\frac{1}{n}\right) = \frac{1}{n} \to 0 \quad n \to \infty$$

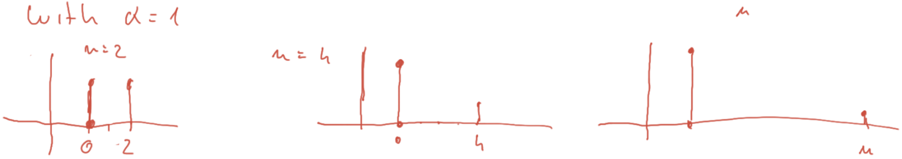
$Z_n \to Z$ in mean square

Exercise 8.8

Suppose that Z_n takes the value n^α with $P(Z_n = n^\alpha) = \frac{1}{n}$

and 0 with $P(Z_n = 0) = 1 - \frac{1}{n}$

with $\alpha = 1$



take $Z = 0$ $\quad E[(Z_n - Z)^2] = (n^\alpha - 0)^2 P(Z_n = n^\alpha) + (0-0)^2 P(Z_n = 0)$

$= n^{2\alpha} \frac{1}{n} = n^{2\alpha - 1} \longrightarrow 0$ only if $2\alpha - 1 < 0 \iff \alpha < \frac{1}{2}$

Theorem: mean-square law of large numbers Let $X_1, X_2, \ldots$ be a sequence of independent random variables, each with mean μ and variance σ^2. The average of the first n of the X_i satisfies as $n \to \infty$

$$\frac{1}{n}(X_1 + X_2 + \cdots + X_n) \to \mu \quad \text{in mean square}$$

Proof

Consider $S_n = X_1 + X_2 + \cdots X_n$

$$E\left[\frac{1}{n} S_n\right] = \frac{1}{n} E(X_1 + X_2 + \cdots X_n) = \frac{1}{n} \cdot n\mu = \mu$$

Then

$$E\left[\left(\frac{1}{n} S_n - \mu\right)^2\right] = var\left(\frac{1}{n} S_n\right) = \frac{1}{n^2} var(X_1 + X_2 + \cdots X_n)$$

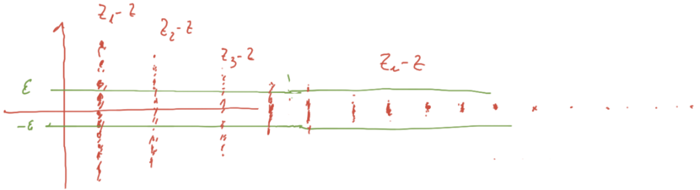
$$= \frac{1}{n^2}\left[var(X_1) + var(X_2) + \cdots var(X_n)\right] = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n} \xrightarrow[n \to \infty]{} 0$$

Convergence in probability

Definition We say that the sequence $Z_1, Z_2, \ldots$ of random variables converges in probability to the (limit) random variable Z if $\forall \epsilon > 0$

$$P(|Z_n - Z| > \epsilon) \to 0 \quad \text{as } n \to \infty$$

If this holds we write $Z_n \to Z$ in probability as $n \to \infty$



Example let $X_1 \dots X_n$ indipendent Uniform $(0,1)$

Take $Z_n = \max\{X_1 \dots X_n\}$ $\quad \max\{X_1 \dots X_n\} \le z \iff X_1 \le z, \dots X_n \le z$

$$F_{Z_n}(z) = P\{Z_n \le z\} = P\{X_1 \le z, X_2 \le z \dots X_n \le z\} = P(X_1 \le z)\, P(X_2 \le z) \cdots P(X_n \le z)$$

$$= P(X_1 \le z)^n = z^n$$

$|a| < \varepsilon \quad -\varepsilon < a < \varepsilon$

$|a| > \varepsilon \quad a > \varepsilon$

$a < -\varepsilon$

Consider the r.v. Z concentrated in 1

$$P\{|Z_n - Z| > \varepsilon\} = P\{|Z_n - 1| > \varepsilon\} = P\{(Z_n - 1) > \varepsilon \cup (Z_n - 1) < -\varepsilon\}$$

$$= P\{Z_n - 1 < -\varepsilon\} = P\{Z_n < 1 - \varepsilon\} = (1-\varepsilon)^n \to 0 \quad \forall \varepsilon > 0$$

$Z_n \xrightarrow{P} 1 \Rightarrow$ try by yourself to check if

$Z_n \to Z = 1$ in mean square

Theorem: Chebyshev's inequality If Y is a random variable and $E(Y^2) < \infty$ then

$$P(|Y| \geq t) \leq \frac{1}{t^2} E(Y^2)$$

Proof Note that $P\{|Y| \geq t\} = P\{Y^2 \geq t^2\}$

Since Y^2 is a positive random variable we can say that

$$P\{|Y| \geq t\} = P\{Y^2 \geq t^2\} \leq \frac{E[Y^2]}{t^2}$$

Theorem If $Z_1, Z_2, \ldots$ is a sequence of random variables and $Z_n \to Z$ in mean square as $n \to \infty$, then $Z_n \to Z$ in probability also

Proof

$$P\{|Z_n - Z| \geq \varepsilon\} \leq \frac{E\{(Z_n - Z)^2\}}{\varepsilon^2}$$

Hence if $E[(Z_n - Z)^2] \to 0$ (for the mean square convergence)

also $P\{|Z_n - Z| \geq \varepsilon\} \to 0$

The converse is false : convergence in probability $\not\Rightarrow$ convergence in mean square

Example 8.19 $Z_n \in \{0, n\}$ $P(Z_n = 0) = 1 - n^{-1}$, $P(Z_n = n) = n^{-1}$. From Ex. 8.8 we know that it does not converge to 0 in mean square ($\alpha = 1$)....but

$$P(|Z_n - 0| > \epsilon) = P(Z_n = n) = \frac{1}{n} \to 0 \quad n \to \infty$$

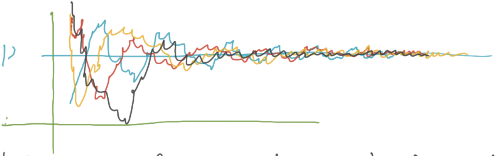
Then Z_n converges to 0 in probability

Weak law of large numbers

Let $X_1, X_2 \dots$ be a sequence of independent random variables each with mean μ and variance σ^2.

The averages $\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \dots X_n)$ of the first n X_i satisfy, as $n \to \infty$

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \dots X_n) \to \mu \quad \text{in probability}$$



(Note that the weak law can be proved also under mild conditions)

- The weak law can be proved without the assumption that the X_i have finite variance...as long as they have the same distribution...
- However the X_i must have a mean. For example if X_i is Cauchy, $\frac{1}{n}(X_1 + \cdots + X_n)$ does not converge to a constant .... in fact it is still Cauchy!
- There are also other laws of large numbers which state stronger form of convergence: *almost sure convergence*

Central limit theorem

- Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with mean μ and varaince σ^2
- By the law of large numbers we know that $S_n = X_1 + \cdots + X_n$ is of order n, in fact $\frac{S_n}{n} \xrightarrow{p} \mu$
- Can we say something about the order of $S_n - n\mu$ and on the *standardized distribution* of S_n, that is

$$Z_n = \frac{S_n - E(S_n)}{\sqrt{var(S_n)}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

- We will show that when $n \to \infty$ Z_n has a distribution *(so the order of is $\sqrt{n}$)* and this distribution is the standard Normal

Theorem: central limit theorem Let $X_1, X_2, \ldots$ be idependent and identically distributed random variables with mean μ and variance $\sigma^2 > 0$. Consider

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

where $S_n = X_1 + X_2 + \cdots X_n$. When $n \to \infty$ we have

$$P(Z_n \leq x) \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \quad for\ x \in \mathbb{R}$$

The distribution of Z_n converges $\forall x$ to the $N(0,1)$ distribution

Proof
For the proof we need the following theorem (that we do not prove)

Continuity theorem. Let $Z_1, Z_2, \ldots,$ be a sequence of random variables with moment generating functions $M_1, M_2 \ldots,$ and suppose that, as $n \to \infty$

$$M_n(t) \to e^{\frac{1}{2}t^2} \quad \textit{for } t \in \mathbb{R}$$

then

$$P(Z_n \leq x) \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \quad \textit{for } x \in \mathbb{R}$$

The disttribution function of Z_n converges to the distribution function of the normal distribution if the moment generating function of Z_n converges to the moment generating function of the normal distribution

Consider

$$U_i = X_i - \mu$$

and note that

$$E(U_i) = 0 \quad E(U_i^2) = var(U_i) = \sigma^2$$

Take

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} U_i$$

The moment generating function of Z_n is

$$\begin{aligned} M_{Z_n}(t) &= E(e^{tZ_n}) = E\left(e^{t\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n} U_i}\right) \\ &= M_{U_1}\left(\frac{t}{\sigma\sqrt{n}}\right) M_{U_1}\left(\frac{t}{\sigma\sqrt{n}}\right) \cdots M_{U_n}\left(\frac{t}{\sigma\sqrt{n}}\right) \\ &= \left[M_U\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n \quad \textit{where } U \stackrel{d}{=} X_i - \mu \end{aligned}$$

there is a typo in formula 8.29 of the book

Now consider the exapnsion of the generating function $M_U(x)$ as power series about $x = 0$

$$\begin{aligned} M_U(x) &= M_U(0) + x\, M^{(1)}(0) + \frac{1}{2}x^2 M^{(2)}(0) + o(x^2) \\ &= 1 + x \cdot 0 + \frac{1}{2}\sigma^2 x^2 + o(x^2) = 1 + \frac{1}{2}\sigma^2 x^2 + o(x^2) \end{aligned}$$

Then by considering $x = \frac{t}{\sigma\sqrt{n}}$ (which when n is large and t is fixed is approximately 0)

$$M_U\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{1}{2}\frac{t^2}{n} + o\left(\frac{1}{n}\right)$$

Hence

$$M_{Z_n}(t) = \left[M_U\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n = \left[1 + \frac{1}{2}\frac{t^2}{n} + o\left(\frac{1}{n}\right)\right]^n \to e^{\frac{1}{2}t^2} \quad \textit{as } n \to \infty$$

- In the proof of the central limit theorem we had $o(x^2)$ which is a function $h(x)$ which converges to 0 faster than x^2 (when $x \to 0$), that is $h(x) = o(x^2)$ if

$$\lim_{x\to 0} \frac{h(x)}{x^2} = 0 \quad (*)$$

- Note also that if $\lim_{x\to x_0} f(x) = L$, then for every sequence a_n such that $\lim_{n\to\infty} a_n = x_0$, $\lim_{n\to\infty} f(a_n) = L$
- In the theorem, the function $h(x) = o(x^2)$ is then evaluated in the point $x = \frac{t}{\sigma\sqrt{n}}$. When t is fixed and as a function of n note that

$$\lim_{n\to\infty} \frac{h(\frac{t}{\sigma\sqrt{n}})}{\frac{t^2}{\sigma^2 n}} = \lim_{n\to\infty} \frac{h(\frac{t}{\sigma}\frac{1}{\sqrt{n}})}{\left(\frac{t}{\sigma\sqrt{n}}\right)^2} = 0$$

(the sequence $a_n = \frac{t}{\sigma\sqrt{n}} \to 0$ and by (*) $\lim_{n\to\infty} h(a_n)/a_n^2 = 0$)

- Hence $h(\frac{t}{\sigma\sqrt{n}})$ (as a function of n) is $o(n^{-1})$ because (when $n \to \infty$) it goes to 0 faster that n^{-1}

Definition The sequence $Z_1, Z_2, \ldots$ is said to converge in distribution to Z as $n \to \infty$ if

$$P(Z_n \leq x) \to P(Z \leq x) \quad \textit{for}\, x \in C$$

where C is the set of reals at which the distribution function $F_Z(z) = P(Z \leq z)$ is continuous

Theorem If $Z_1, Z_2, \ldots$ is a sequence of random variables and Z_n converges to Z in probability, then Z_n converges to Z in distribution.

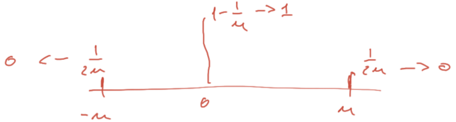
The converse is generally false, unless the convergence is to a constant c

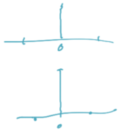
Theorem If $Z_1, Z_2, \ldots$ is a sequence of random variables and Z_n converges to a constant c in distribution, then Z_n converges to c in probability also

Consider the random variable

$$X_n \sim \{-n, 0, n\}$$

$$P\{X=0\} = \frac{n-1}{n} \qquad P\{X=-n\} = P\{X=n\} = \frac{1}{2n}$$





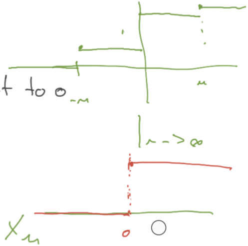
$$E(X_n) = \frac{1}{2n}(-n) + 0\,\frac{n-1}{n} + n\,\frac{1}{2n} = 0$$

$$E(X_n^2) = (-n)^2 \cdot \frac{1}{2n} + 0^2\,\frac{n-1}{n} + n^2\,\frac{1}{2n} = n^2\,\frac{1}{2n} + n^2 \cdot \frac{1}{2n} = n$$

$$VAR(X_n) = E(X_n^2) - E(X_n)^2 = n - 0 = n$$

Does the sequence of random variables converge in mean square to 0?

$$E\left(\left(X_n - 0\right)^2\right) = Var(X_n) = n \to \infty \quad \text{not to } 0$$



There is not convergence in mean square

Calculate the distribution function of X_n

$$F(x) = P(X_n \leq x) = \begin{cases} 0 & x < -n \\ \frac{1}{2n} & -n \leq x < 0 \\ \frac{1}{2n} + 1 - \frac{1}{n} & 0 \leq x < n \\ \frac{1}{2n} + 1 - \frac{1}{n} + \frac{1}{2n} & n \leq x < \infty \end{cases} = \begin{cases} 0 & x < -n \\ \frac{1}{2n} & -n \leq x < 0 \\ 1 - \frac{1}{2n} & 0 \leq x < n \\ 1 & n \leq x < \infty \end{cases}$$