Stochastic Processes

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Week 11

Theorem The probability that a simple random walk ever revisits its starting point is given by

$$P(S_n = 0 \text{ for some } n > 0 | S_0 = 0) = 1 - |p - q|$$

Hence the walk is recurrent if and only if $p = q$

Proof. Suppose that $S_0 = 0$ and let

$$A_n = \{S_n = 0\}$$

be the event that the walk is in 0 at time n and

$$B_n = \{S_n = 0, S_k \neq 0 \text{ for } 1 \leq k \leq n-1\}$$

be the event that the *first return* is a time n. Note that

$$A_n \subset \bigcup_{k=1}^{n} B_k,$$

in fact if at time n the walk is in 0, the first return occurred at time $k \leq n$

Then, by observing that $B_k \cap B_l = \emptyset$,

$$P(A_n) = P\left(A_n \cap \bigcup_{k=1}^{n} B_n\right) = \sum_{k=1}^{n} P(A_n \cap B_k)$$

Since transitions in disjoint intervals of time are independent of each, $P(A_n|B_k) = P(A_n|S_k = 0) = P(A_{n-k})$, hence

$$P(A_n \cap B_k) = P(A_n|B_k)P(B_k) = P(A_{n-k})P(B_k) \quad 1 \leq k \leq n$$

Thus we can write

$$P(A_n) = \sum_{k=1}^{n} P(B_k)P(A_{n-k})$$

Writing $u_n = P(A_n)$ and $f_n = P(B_n)$

$$u_n = \sum_{k=1}^{n} f_k u_{n-k} \quad \mathit{for}\ n = 1, 2, \ldots$$

Now consider the generating functions of $u_0, u_1, \dots$ and $f_0, f_1, \dots$

$$U(s) = \sum_{n=0}^{\infty} u_n s^n \quad F(s) = \sum_{n=0}^{\infty} f_n s^n$$

where $u_0 = P(A_0) = P(S_0 = 0) = 1$ since the walk starts in 0 and $f_0 = P(B_0) = 0$ since the initial position cannot be considered a first return

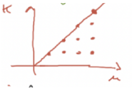
If $|s| < 1$, $|u_n| \leq 1$ and $|f_n| \leq 1$ (since they are probabilities) then

$$\sum_{n=0}^{\infty} |u_n s^n| \leq \sum_{n=1}^{\infty} |s^n| = \sum_{n=1}^{\infty} |s|^n < \infty \quad \textit{for } |s| < 1$$

$$\sum_{n=0}^{\infty} |f_n s^n| \leq \sum_{n=1}^{\infty} |s^n| = \sum_{n=1}^{\infty} |s|^n < \infty \quad \textit{for } |s| < 1$$

Hence $U(s)$ and $F(s)$ converge absolutely If $|s| < 1$, Then, if $|s| < 1$

$$U(s) - 1 = \sum_{n=1}^{\infty} u_n s_n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_k u_{n-k} s^n =$$



$$= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} f_k u_{n-k} s^k s^{n-k} = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} f_k u_m s^k s^m$$

$$= \sum_{k=1}^{\infty} f_k s^k \sum_{m=0}^{\infty} u_m s^m = \sum_{k=0}^{\infty} f_k s^k \sum_{m=0}^{\infty} u_m s^m = F(s)U(s)$$

So we have, if $|s| < 1$,

$$F(s) = \frac{U(s) - 1}{U(s)} = 1 - \frac{1}{U(s)}$$

By the identity $\binom{2m}{m} = \binom{-1/2}{m}(-4)^m$ and the extended binomial theorem $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$

$$\begin{aligned} U(s) &= \sum_{n=0}^{\infty} u_n s^n = \sum_{m=0}^{\infty} u_{2m} s^{2m} = \sum_{m=0}^{\infty} \binom{2m}{m} p^m q^m (s^2)^m \\ &= \sum_{m=0}^{\infty} \binom{-1/2}{m} (-4)^m p^m q^m (s^2)^m = (1-4pqs^2)^{-1/2} \quad |s|<1 \end{aligned}$$

$$F(s) = 1 - \frac{1}{U(s)} = 1 - (1-4pqs^2)^{1/2} \quad |s|<1$$

Finally $P(S_n = 0 \text{ for some } n \geq 1 \mid S_0 = 0) = P(B_1 \cup B_2 \cup \cdots)$

$$= \sum_{n=1}^{\infty} P(B_n) = \sum_{n=1}^{\infty} f_n = \lim_{s \uparrow 1} \sum_{n=1}^{\infty} f_n s^n = \lim_{s \uparrow 1} F(s) = F(1) =$$

$$= 1 - \sqrt{1-4pq} = 1 - \sqrt{(p+q)^2 - 4pq} = 1 - \sqrt{p^2+q^2+2pq-4pq}$$

$$= 1 - \sqrt{(p-q)^2} = 1 - |p-q|$$

Expectation for the first return time

- Let

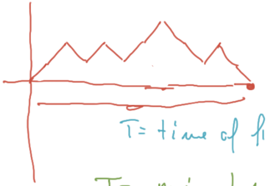
$$T = \min\{n \geq 1 : S_n = 0\}$$

  be the random time until the first return
- If $p \neq q$ $P(T = \infty) > 0$ then $E(T) = \infty$ (think that we have $\infty * P(T = \infty)$ in the sum for the expectation)
- If $p = q$, $P(T < \infty) = 1$ but

$$E(T) \;=\; \sum_{n=1}^{\infty} n f_n = \infty \quad \text{(see next slide)}$$

- Although a symmetric random walk is certain to return to its starting point, the expected value of the time which elapses before this happens is infinite.

The time of the first return



T = time of first return to 0

$$T = \min\{n \geq 1 : S_n = 0\}$$

$$\{T < \infty\} \Leftrightarrow \{S_n = 0 \text{ for some } n = 1, 2, \dots\} = \{\exists n \geq 1 : S_n = 0\}$$

Hence if $p = q$: $1 = P\{S_n = 0 \text{ for some } n = 1, 2 \dots\} = P\{T < \infty\}$.

if $p=q$ $E(T)=\infty$

Note that $T=n$ is exactly the event $B_n = \{S_n = 0, S_k \neq 0 \; k=1,2,\dots,n-1\}$

$$P\{T=n\} = P\{B_n\} = f_n$$

$$E(T) = \sum_{n=1}^{\infty} n \cdot P\{T=n\}$$

$$= \sum_{n=1}^{\infty} n f_n = \sum_{n=0}^{\infty} (n+1) f_{n+1}$$

$$= \lim_{s \uparrow 1} \sum_{n=0}^{\infty} (n+1) f_{n+1} s^n$$

$$= \lim_{s \uparrow 1} \sum_{n=1}^{\infty} n f_n s^{n-1}$$

Moreover $F(s) = \sum_{n=0}^{\infty} f_n s^n < \infty \quad |s|<1$

then $\frac{d}{ds} F(s) = \sum_{n=1}^{\infty} n f_n s^{n-1} < \infty \quad |s|<1$

$$= \sum_{n=0}^{\infty} (n+1) f_{n+1} s^n$$

$$= \lim_{s \uparrow 1} F'(s) = \lim_{s \uparrow 1} \frac{d}{ds}\left(1 - \sqrt{1 - 4 \cdot \tfrac{1}{2}\tfrac{1}{2} s^2}\right)$$

$$= \lim_{s \uparrow 1} \frac{d}{ds}\left(1 - \sqrt{1-s^2}\right) = \lim_{s \uparrow 1} \frac{s}{(1-s^2)^{1/2}} = \infty$$

The Gambler's Ruin Problem

- Consider a game between two players, A and B
- A possesses a euro and B possesses $(N - a)$ euro so that their total capital is N euro
- A coin is flipped repeatedly, $p = P(H)$, $q = P(T)$
- Each time heads turns up, B gives 1 euro to A, while each time tails turns up, A gives 1 euro to B.
- This game continues until either A or B runs out of money.
- We record the state of play by noting A's capital

  $$S_n \in \{0, 1, \dots, N\}$$

  after each flip.

- This walk starts at the point a and follows a simple random walk until it reaches either 0 or N, at which time it stops
- We speak of 0 and N as being absorbing barriers since the random walker sticks to whichever of these points it hits first.
- We shall say that A wins the game if the random walker is absorbed at N, and that B wins the game if the walker is absorbed at 0.

Theorem Consider a simple random walk on $\{0, 1, \ldots, N\}$ with absorbing barriers at 0 and N. If the walk begins at the point a, where $0 \leq a \leq N$ then the probability $\nu(a)$ that the walk is absorbed at N is given by

$$\nu(a) = \begin{cases} \dfrac{(q/p)^a - 1}{(q/p)^N - 1} & p \neq q \\ a/N & p = q \end{cases}$$

Proof. Let H be the event that the first flip of the coin shows heads

$$\begin{aligned} \nu(a) &= P(A\,wins) = P(A\,wins|H)P(H) + P(A\,wins|T)P(T) \\ &= \nu(a+1)p + \nu(a-1)q \quad 1 \leq a \leq N-1 \end{aligned}$$

Moreover

$$\nu(0) = 0 \quad \nu(N) = 1$$

Note that

- $\nu(a) = (p+q)\nu(a) = \nu(a+1)p + \nu(a-1)q$,
- $p(\nu(a+1) - \nu(a)) = q(\nu(a) - \nu(a-1))$
- $\nu(a+1) - \nu(a) = \frac{q}{p}(\nu(a) - \nu(a-1))$

Then

$$\begin{aligned}
\nu(2) - \nu(1) &= \frac{q}{p}(\nu(1) - \nu(0)) = \frac{q}{p}\nu(1) \\
\nu(3) - \nu(2) &= \frac{q}{p}(\nu(2) - \nu(1)) = \left(\frac{q}{p}\right)^2 \nu(1) \\
&\vdots \\
\nu(a+1) - \nu(a) &= \frac{q}{p}(\nu(a) - \nu(a-1)) = \left(\frac{q}{p}\right)^a \nu(1)
\end{aligned}$$

$$
\begin{aligned}
\nu(a+1) - v(1) &= \nu(a+1) - \nu(a) + \nu(a) - v(1) \\
&= \nu(a+1) \pm \nu(a) \pm \nu(a-1) \pm \cdots v(2) - v(1) \\
&= \sum_{i=0}^{a} v(i+1) - v(i) = \sum_{i=1}^{a} \left(\frac{q}{p}\right)^{i} v(1)
\end{aligned}
$$

That is if $p \neq q$

$$
\begin{aligned}
\nu(a+1) &= v(1) + \sum_{i=1}^{a} \left(\frac{q}{p}\right)^{i} v(1) = v(1) \sum_{i=0}^{a} \left(\frac{q}{p}\right)^{i} \\
&= v(1) \frac{1-(q/p)^{a+1}}{1-q/p}
\end{aligned}
$$

and if $p = q$

$$
\nu(a+1) = (a+1)v(1)
$$

Now take $a = N - 1$, then if $p \neq q$

$$1 = \nu(N) = \nu(1)\frac{1-(q/p)^N}{1-q/p}$$

Hence

$$\nu(1) = \frac{1-q/p}{1-(q/p)^N}$$

and

$$\nu(a) = \frac{1-(q/p)^a}{1-(q/p)^N} = \frac{(q/p)^a - 1}{(q/p)^N - 1}$$

Similarly, if $p = q$

$$1 = \nu(N) = N\nu(1) \quad \nu(1) = 1/N$$

and

$$\nu(a) = \frac{a}{N}$$

end proof!

By the same argument the probability that B wins the game when her/his capital is $N-a$

$$\mu(a) = \begin{cases} \dfrac{(p/q)^{N-a}-1}{(p/q)^N-1} & p \neq q \\ (N-a)/N & p = q \end{cases}$$

Hence when $p \neq q$

$$\begin{aligned} \nu(a)+\mu(a) &= \frac{(q/p)^a-1}{(q/p)^N-1}+\frac{(p/q)^{N-a}-1}{(p/q)^N-1} \\ &= \frac{(q/p)^a-1}{(q/p)^N-1}+\frac{(q/p)^{a-N}-1}{(q/p)^{-N}-1} \\ &= \text{some algebra...} = \frac{2-x^N-x^{-N}}{2-x^N-x^{-N}} = 1 \end{aligned}$$

and when $p = q$

$$\nu(a)+\mu(a) = \frac{a}{N}+\frac{N-a}{N} = 1$$

Then $P(A\, wins)+P(B\, wins) = 1$ and there is zero probability that the game will fail to terminate.

- Finally, what are A's fortunes if the opponent is infinitely rich?
- In practice, this situation cannot arise, but the hypothetical situation may help us to understand the consequences of a visit to the casino at Monte Carlo.
- In this case, A can never defeat the opponent, (the B's capital is infinite) but A might at least hope to be spared ultimate bankruptcy in order to play the game forever.

Theorem Consider a simple random walk on $\{0, 1, \ldots\}$ with an absorbing barrier in 0. If the walk begins at the point $a \geq 0$ the probability $\pi(a)$ that the walk is ultimately absorbed at 0 is given by

$$\pi(a) = \begin{cases} (q/p)^a & p > q \\ 1 & p \neq q \end{cases}$$

- The probability that player A is able to play forever is strictly positive if and only if the odds are stacked in his or her favour at each flip of the coin.
- An intuitive approach to this theorem is to think of this new game as the limit of the previous game as the total capital N tends to infinity while A's initial capital remains fixed at a,

$$P(A\,is\,bankrupted) = P(B\,wins) = \lim_{N \to \infty} 1 - \nu(a)$$

Note that

- When $p > q$,

$$\lim_{N\to\infty} 1 - \frac{1-(q/p)^a}{1-(q/p)^N} = 1 - (1-(q/p)^a) = (q/p)^a$$

  A has a positive probability to avoid bankrupt even if B's capital tends to infinity

- When $p < q$

$$\lim_{N\to\infty} 1 - \frac{(q/p)^a - 1}{(q/p)^N - 1} = 1 - 0 = 1$$

  A has not chance to avoid bankrupt when B's capital tends to infinity

- When $p = q$

$$\lim_{N\to\infty} 1 - \frac{a}{N} = 1$$

  A has not chance to avoid bankrupt when B's capital tends to infinity

Theorem Consider a simple random walk on $\{0, 1, \ldots\}$ with an absorbing barrier at 0 and N. If the walk begins at the point $a \geq 0$ if the walk begins at a point a, where $0 \leq a \leq N$, then the expected number $e(a)$ of steps before the walk is absorbed at either 0 or N is given by

$$e(a) = \begin{cases} \dfrac{1}{p-q}\left(N\dfrac{(q/p)^a - 1}{(q/p)^N - 1} - a\right) & p \neq q \\ a(N-a) & p = q \end{cases}$$

No proof

Markov chains

- A stochastic process is said to have the 'Markov property' if, conditional on its present value, its future is independent of its past.
- Let S be a countable set and let $\mathbf{X} = (X_n : n \geq 1)$ be a sequence of random variables taking value in S.
- The X_n are functions on the same probability space
- The sequence $\mathbf{X}$ is a Markov chain if, conditional on the present value X_n, the future $(X_r : r > n)$ is independent of the past $(X_m : m < n)$.

Definition The sequence $\mathbf{X}$ is called a Markov chain if it satisfies the Markov property

$$P(X_{n+1} = i_{n+1} | X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_{n+1} = i_{n+1} | X_n = i_n)$$

for al $n > 0$ and all $i_0, i_1, \ldots i_n \in S$.

The Markov chain is called homogeneous if, for all $i, j \in S$, the conditional probability $P(X_{n+1} = j | X_n = i)$ does not depend on the value of n

We will consider only homogeneous Markov chain

Given a Markov chain, we call **transition matrix** , the matrix

$$P = (p_{i,j} : i, j \in S)$$

where $p_{i,j} = P(X_1 = j | X_0 = i) = P(X_{n+1} = j | X_n = i)$.

If $|S| = k$

$$P = \begin{pmatrix} p_{1,1} & \cdots & p_{1,k} \\ \vdots & \vdots & \vdots \\ p_{k,1} & \cdots & p_{1,k} \end{pmatrix}$$

We call $\lambda = (\lambda_i, i \in S)$ where $\lambda_i = P(X_0 = i)$ the **initial distribution**

- The vector λ is a **distribution** in that $\lambda_i \geq 0$ for $i \in S$ and $\sum_{i \in S} \lambda_i = 1$ In fact the λ_i are non negative since are probabilities and
$$\sum_{i \in S} \lambda_i = \sum_{i \in S} P(X_0 = i) = P(X_0 \in S) = 1$$
- The matrix $P = (p_{i,j})$ is a **stochastic matrix** in that
  (i) $p_{i,j} \geq 0$ for $i, j \in S$
  (ii) $\sum_{j \in S} p_{i,j} = 1$ for $\imath \in S$ so that P has row sums 1 In fact $p_{i,j} \geq 0$ since it is a probability and
$$\sum_{j \in S} p_{i,j} = \sum_{j \in S} P(X_1 = j | X_0 = i) = P(X_1 \in S | X_0 = i) = 1$$

Theorem Let λ be a distribution and P a stochastic matrix. The random sequence $\mathbf{X} = (X_n : n \geq 0)$ is a Markov chain with initial distribution λ and transition matrix P if and only if

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = \lambda_{i0} p_{i0,i_1} \cdots p_{i_{n-1},i_n} \quad (*)$$

for all $n \geq 0$ and $i_0, i_1, \ldots, i_n \in S$

Proof. Markov chain $\Rightarrow (*)$ Let A_k be the event $\{X_k = i_k\}$. The condition (*) can be written as

$$P(A_0 \cap A_1 \cap \cdots \cap A_n) = \lambda_{i0} p_{i0,i_1} \cdots p_{i_{n-1},i_n}$$

Let $\mathbf{X}$ be a Markov chain with initial distribution λ and transition matrix P. Suppose that the condition (*) is true for $n < N$ then

$$\begin{aligned} P(A_0 \cap A_1 \cap \cdots \cap A_N) &= P(A_0 \cap A_1 \cap \cdots \cap A_{N-1}) P(A_N | A_0, \ldots, A_{N-1}) \\ &= P(A_0 \cap A_1 \cap \cdots \cap A_{N-1}) P(A_N | A_{N-1}) \\ &= \lambda_{i0} p_{i0,i_1} \cdots p_{i_{N-2},i_{N-1}} \, p_{i_{N-1},i_N} \end{aligned}$$

Proof. (*) $\Rightarrow$ *Markov chain* Suppose that the condition holds for all n and sequence (i_m). For $n = 0$ we obtain $P(X_0 = i_0) = \lambda_{i_0}$. To prove the Markov property note that

$$\begin{aligned} p(A_{n+1}|A_0 \cap A_1 \cap \cdots, A_n) &= \frac{P(A_0 \cap A_1 \cap \cdots \cap A_{n+1})}{P(A_0 \cap A_1 \cap \cdots \cap A_n)} \\ &= \frac{\lambda_{i0} p_{i0,i_1} \cdots p_{i_{n-1},i_n} p_{i_n,i_{n+1}}}{\lambda_{i0} p_{i0,i_1} \cdots p_{i_{n-1},i_n}} \\ &= p_{i_n,i_{n+1}} \end{aligned}$$

Since this does not depend on the states $i_0, i_1, \ldots, i_{n-1}$ $\mathbf{X}$ is a homogeneous Markov chain with initial distribution λ e transition matrix P

Theorem Let $\mathbf{X}$ be a Markov chain. For $n \geq 0$, for any event H given in terms of past history $X_0, X_1, \ldots X_{n-1}$ and any events F given in terms of the future $X_{n+1}, X_{n+2}, \ldots$

$$P(F|X_n = i, H) = P(F|X_n = i)$$

Proof
Indicate with $\sum_{<n}$ the sum over all sequence of states $(i_0, i_1, \ldots i_{n-1})$ corresponding to the event H and with $\sum_{>n}$ the sum over all sequence of states $(i_{n+1}, i_{n+2}, \cdots)$ corresponding to the event F

$$\begin{aligned}
P(F|X_n = i, H) &= \frac{P(H, X_n = i, F)}{P(H, X_n = i)} \\
&= \frac{\sum_{<n} \sum_{>n} p_{i0,i_1} \cdots p_{i_{n-1},i} p_{i,i_{n+1}} p_{i_{n+1},i_{n+2}} \cdots}{\sum_{<n} p_{i0,i_1} \cdots p_{i_{n-1},i}} \\
&= \frac{\sum_{<n} p_{i0,i_1} \cdots p_{i_{n-1},i} \sum_{>n} p_{i,i_{n+1}} p_{i_{n+1},i_{n+2}} \cdots}{\sum_{<n} p_{i0,i_1} \cdots p_{i_{n-1},i}} \\
&= \sum_{>n} p_{i,i_{n+1}} p_{i_{n+1},i_{n+2}} \cdots = P(F|X_n = i)
\end{aligned}$$

Transition probabilities

The $n-$step transition probability is given by

$$p_{i,j}(n) = P(X_n = j | X_0 = i) \quad i, j \in S$$

The n-step transition matrix is the matrix $P(n)$ given by the n-step transition probabilities

Note that

$$\begin{aligned} p_{ij}(2) &= P(X_2 = j | X_0 = i) = \sum_{k \in S} P(X_2 = j, X_1 = k | X_0 = i) \\ &= \sum_{k \in S} P(X_2 = j | X_1 = k, X_0 = i) P(X_1 = k | X_0 = i) \\ &= \sum_{k \in S} P(X_2 = j | X_1 = k) P(X_1 = k | X_0 = i) = \sum_{k \in S} p_{i,k} p_{k,j} \end{aligned}$$

Hence $p_{ij}(2)$ is the product between the $i-$th row of P and the $j-$th column of P, hence $P(2) = PP$... but we have a more general result

Chapman-Kolmogorov equation. We have that

$$p_{i,j}(m+n) = \sum_{k \in S} p_{i,k}(m) p_{k,j}(n)$$

for $i, j \in S$ and $m, n \geq 0$. That is to say that $P(m+n) = P(m)P(n)$

Proof

$$\begin{aligned}
p_{i,j}(m+n) &= P(X_{m+n} = j | X_0 = i) \\
&= \sum_{k \in S} P(X_{m+n} = j, X_m = k | X_0 = i) \\
&= \sum_{k \in S} P(X_{m+n} = j | X_m = k, X_0 = i) P(X_m = k | X_0 = i) \\
&= \sum_{k \in S} P(X_{m+n} = j | X_m = k,) P(X_m = k | X_0 = i) \\
&= \sum_{k \in S} p_{i,k}(m) p_{k,j}(n)
\end{aligned}$$

Note that $P(1) = P$ and

$$P(n) = P(n-1+1) = P(n-1)P(1) = P(n-1)P = P(n-2)PP = \cdots = P^n$$

One way of calculating the n step transition probabilities is therefore to find the nth power of the matrix P