Stochastic Processes

Andrea Tancredi

Sapienza University of Rome

Week 13

Invariant distributions for finite Markov chain

Exercises

Invariant distribution for finite Markov chains

- Let X_n be a Markov chain with finite state space $S = \{1, \dots, k\}$ and transition matrix P. The invariant distribution is the (row) vector $\pi = (\pi_1, \dots, \pi_k)$ such that $\pi P = \pi$ and $\sum_{j=1}^{k} \pi_j = 1$ with $p_i \geq 0$ for all $i = 1, \dots, k$.
- To find the invariant distribution we need to solve the system of $k+1$ equations with k unknowns where the the first k equations can be written as the homogeneous system (in row form)

$$\pi(P - I) = (0, \dots, 0)$$

  where I is the identity matrix and the other equation is

$$\pi(1, \cdots, 1)^t = 1$$

- An important point is to understand if this system has a solution and if this solution is unique.

- Note that the rank of $P - I$ is less than k ($det(P - I) = 0$). In fact the rows of P sum to 1, hence the row of $P - I$ sum to 0 which means the the colums of $P - I$ are linearly dependent.
- Thus the homogeneous system $\pi(P - I) = (0, \ldots, 0)$ has infinite solutions (as well as the equivalent system $(P - I)^t \pi^t = (0, \ldots, 0)^t$)
- Specifically, the space of solutions of $\pi(P - I) = (0, \ldots, 0)$ is a subspace of $\mathbb{R}^k$ of dimension $k - rank(P - I)$
- It will be crucial to have a positive solution, that is a solution with all elements $\pi_i \geq 0$. In fact if all the solutions of the sytem $\pi P = \pi$ will have both positive and negative signs we will never find a probability distribution

- We now prove that if P is irreducible then there exist a positive solution π *This proof has not been done during the course and it is not part of the program for the oral exam*

- In fact suppose that ν is such that $\nu P = \nu$ and take the matrix $Q = (P + I)/2$ and $R = Q^{k-1}$. Note that Q also is a transition matrix of a chain that with probability $1/2$ behaves like P and that ν is such that $\nu Q = \nu$ and $\nu R = \nu$ and

- Note also that $R_{i,j} > 0$ for all i, j. In fact P is irreducible and for any $i \neq j$ there is a path from i to j. Since the shortest such path will not visit any state more than once, we can always get from i to j in $k - 1$ steps, and it follows that $R_{i,j} > 0$

- Suppose that $\nu_i < 0$ while all the other elements are positive. Then

$$|\nu_j| = |\sum_{\ell=1}^{k} \nu_\ell R_{\ell,j}| = |\nu_1 R_{1,j} + \dots \nu_i R_{i,j} + \dots \nu_k R_{k,j}| < \sum_{\ell=1}^{k} |\nu_\ell| R_{\ell,j}$$

  Hence we have the impossible inequality

$$\sum_{j=1}^{k} |\nu_j| < \sum_{j=1}^{k} \sum_{\ell=1}^{k} |\nu_\ell| R_{\ell,j} = \sum_{\ell=1}^{k} |\nu_\ell| \sum_{j=1}^{k} R_{\ell,j} = \sum_{\ell=1}^{k} |\nu_\ell|$$

- Note that $\nu_i > 0$ for $i = 1, \dots, k$. In fact we know that $\nu_i \geq 0$ for $i = 1, \dots, k$ with at least one element strictly positive (ν is differet from the trival solution $(0, \dots, 0)$) and $R_{i,j} > 0$ then

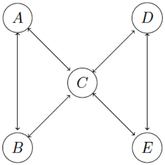
$$\nu_i = \sum_{j=1}^{k} \nu_j R_{i,j} > 0 \quad \textit{and} \quad \pi = \frac{1}{\sum_{j=1}^{k} \nu_j} \nu$$

  solve the system $\pi P = \pi$ and also the equation $\pi(1, \cdots, 1)^t = 1$. Hence if P is irreducible (and finite) an invariant distribution exists.

- The uniqueness can be seen by observing that if $\exists\, \psi \neq \pi$ such that $\psi P = \psi$ where ψ is not proportional to ν, the space of solutions of the system $\pi P = \pi$ would be a linear subspace of $\mathbb{R}^k$ of dimension 2
- Then also solutions with positive and negative terms would be admitted, (think that every vector $a\nu + b\psi$ would solve $\pi P = \pi$) which is impossible when P is irreducible

- In conclusion, when the chain is finite and P irreducible the invariant distribution exists and it is also unique (and the chain will be positive recurrent)
- We can find the invariant distribution by removing one equation from the system $\pi P = \pi$ but considering the equation $\pi(1, \ldots, 1)^t = 1$. This way we have a non homogeneous linear system of k equations and k unknowns.
- Moreover, If the chain is aperiodic the invariant distribution will be also the limit distribution of the chain

Exercise. Written exam July 2021 A particle follows a random walk on the set described in the following figure



At time *n*, the particle moves to one of the adjacent nodes with equal probability

(a) Write the transition matrix of the chain

(b) Find the invariant distribution

(c) Is the chain ergodic and aperiodic?

The transition matrix is

$$P = \begin{pmatrix} & A & B & C & D & E \\ A & 0 & 1/2 & 1/2 & 0 & 0 \\ B & 1/2 & 0 & 1/2 & 0 & 0 \\ C & 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ D & 0 & 0 & 1/2 & 0 & 1/2 \\ E & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

The invariant distribution is the (row) vector $\pi = (\pi_A, \pi_B, \pi_C, \pi_D, \pi_E)$ such that $\pi P = \pi$ and $\sum_{j=A}^{E} \pi_j = 1$ which can be obtained by solving

$$\begin{cases} \frac{1}{2}\pi_B + \frac{1}{4}\pi_C & = \pi_A \\ \frac{1}{2}\pi_A + \frac{1}{4}\pi_C & = \pi_B \\ \frac{1}{4}\pi_C + \frac{1}{2}\pi_E & = \pi_D \\ \frac{1}{4}\pi_C + \frac{1}{2}\pi_D & = \pi_E \\ \pi_A + \pi_B + \pi_C + \pi_D + \pi_E & = 1 \end{cases}$$

From the first two equations we obtain $\pi_B - \pi_A = 2(\pi_A - \pi_B)$, that is $\pi_A = \pi_B$ so that (look the first) $\pi_C = 2\pi_A$. From the third and fourth equations we obtain $\pi_E - \pi_D = 2(\pi_D - \pi_E)$, that is $\pi_D = \pi_E$ so that (look the third) $\pi_C = 2\pi_D$ which means $\pi_A = \pi_D$. Then from the last equation is $6\pi_A = 1$ and the invariant is $\pi = (1/6, 1/6, 1/3, 1/6, 1/6)$

Since all the states communicate the chain is irreducible. Irreducible finite chains have always the invariant distribution (which is unique) hence they are positive recurrent.

The chain is also aperiodic since $p_{AA}(2) > 0$ and $p_{AA}(3) > 0$ and the period is a class property.

Since the chain is ergodic and aperiodic the invariant distribution is also the limit distribution of the chain.

Exercise. Written exam October 2021 Three children, A, B, and C throw a ball between them. When A has the ball, he throws it to B with probability 0.2 or to C with complementary probability. When B has the ball, he throws it to A with probability 0.6 or to C with complementary probability. Finally, when C has the ball, he throws it to A or B with the same probability. Let X_n be the children throwing the ball at time $n = 0, 1, \ldots$

(a) Consider the process as a Markov chain and write the transition matrix of the process.(see slides week 12)

(b) Suppose that the children throwing the ball at time 0 is uniformly selected, who has the higher probability of throwing the ball when $n = 2$.(see slides week 12)

(c) Answer to the same question when $n = 100$

The transition matrix is

$$P = \begin{pmatrix} 0 & 0.2 & 0.8 \\ 0.6 & 0 & 0.4 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

For the point (c) we need to see if the chain is irreducible and aperiodic. In fact finite aperiodic and irreducible chains converge to the invariant distribution. The three states represent a communicating class, hence the chain is irreducible. The period is 1, in fact $p_{AA}(2) > 0$ and $p_{AA}(3) > 0$. Then the chain is irreducible and aperiodic.

The limit distribution is the invariant which solves the system

$$\begin{cases} 6\pi_B + 5\pi_C = 10\pi_A \\ 2\pi_A + 5\pi_C = 10\pi_B \\ \pi_A + \pi_B + \pi_C = 1 \end{cases}$$

From the first two equations we have $6\pi_B - 2\pi_A = 10(\pi_A - \pi_B)$ which is $\pi_A = \frac{4}{3}\pi_B$ and from the first we obtain $18\pi_B + 15\pi_C = 40\pi_B$ that is $\pi_B = \frac{15}{22}\pi_C$. The last equation is then $\pi_B(\frac{4}{3} + 1 + \frac{22}{15}) = 1$, that is $\pi_B = \frac{20+15+22}{15} = 1$, hence $\pi_B = \frac{15}{57}$, $\pi_A = \frac{20}{57}$ and $\pi_C = \frac{22}{57}$

In the long run (and approximately also when $n = 100$) C has the higher probability of throwing the ball

Exercise. Let X_n be a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

Suppose that the initial distribution is $\lambda = (1/2, 1/2, 0)$

(a) Find $P(X_0 = 1, X_2 = 3)$, $P(X_0 + X_2 = 4)$ and $P(X_0 = 1, X_2 = 3 | X_0 + X_2 = 4)$

(b) Is the chain ergodic? In case, find the invariant distribution

(c) Find

$$\lim_{n \to \infty} P(X_n + X_{n+2} = 4)$$

$$
\begin{aligned}
P(X_0 = 1, X_2 = 3) &= \sum_{j=1}^{3} P(X_0 = 1, X_1 = j, X_2 = 3) = P(X_0 = 1, X_1 = 2, X_2 = 3) \\
&= P(X_0 = 1)P(X_1 = 2|X_0 = 1)P(X_2 = 3|X_2 = 2) = \frac{1}{2}\frac{2}{3}\frac{2}{3} = \frac{2}{9}
\end{aligned}
$$

$$
\begin{aligned}
P(X_0 + X_2 = 4) &= \sum_{j=1}^{3} P(X_0 + X_2 = 4|X_0 = j)P(X_0 = j) \\
&= P(X_2 = 4 - 1|X_0 = 1)P(X_0 = 1) + P(X_2 = 4 - 2|X_0 = 2)P(X_0 = 2) \\
&= \frac{1}{2}\frac{2}{3}\frac{2}{3} + \frac{1}{2}\left(\frac{1}{3}\frac{2}{3} + \frac{2}{3}\frac{1}{3}\right) = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}
\end{aligned}
$$

$$
P(X_0 = 1, X_2 = 3|X_0 + X_2 = 4) = \frac{1}{2}
$$

It is ergodic and the invariant is $\pi = (1/3, 1/3, 1/3)$. For the invariant distribution note that the transition matrix is double stochastic, that is also the columns sum to 1. In this cases it easy to see that the uniform distribution is the invariant one.

Finally note that

$$\begin{aligned} P(X_n + X_{n+2} = 4) &= \sum_{j=1}^{3} P(X_n + X_{n+2} = 4 | X_n = j) P(X_n = j) \\ &\approx \frac{1}{3}\left(\frac{4}{9} + \frac{4}{9} + \frac{1}{9}\right) = \frac{1}{3} \end{aligned}$$

Exercise Let X, Y be a continuous random variable with density

$$f_{XY}(x, y) = 12xy(1-y) \quad 0 < x < 1, 0 < y < 1$$

1. Calculate the covariance of X, Y. 0
2. Consider the transformation $S = X$ and $T = XY^2$. Find the set of values where the density of S, T is positive.
   $A = \{s, t : 0 < s < 1, 0 < t < s\}$
3. Find the joint density of S, T. $f_{ST}(s, t) = 6(1 - \sqrt{t/s})$
4. Find the density of T. $f_T(t) = 6t - 12\sqrt{t} + 6$

Exercise A smoker has two match boxes where there are 5 matches. Every time the smoker lights up a cigarette he takes a match randomly from one of the boxes and he throws it away. Let X be the number of matches in the non empty box when the first box is empty. Write the probability mass function of X

Note $X \in \{1, 2, 3, 4, 5\}$. To have $X = k$ we have smoked 4 cigarettes to have a box with only a match and 5-k cigarettes to have the other with exactly k matches. Finally we need to take a match from the box with only one match. Then

$$
\begin{aligned}
P(X = k) &= 2\binom{4+5-k}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{4+5-k-4}\frac{1}{2} \\
&= \binom{9-k}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-k}
\end{aligned}
$$

Observe that

$$
\begin{aligned}
P(X=1) &= \binom{8}{4}\frac{1}{2^8} = 0.2734 \\
P(X=2) &= \binom{7}{4}\frac{1}{2^7} = 0.2734 \\
P(X=3) &= \binom{6}{4}\frac{1}{2^6} = 0.2344 \\
P(X=4) &= \binom{5}{4}\frac{1}{2^5} = 0.1562 \\
P(X=5) &= \binom{4}{4}\frac{1}{2^4} = 0.0625
\end{aligned}
$$

and the probabilities sum to 1

Exercise. Let X and Y be independent and Uniform on (0,1). Consider

$$U = |X - Y| \quad V = \min\{X, Y\}$$

(a) Find the marginal density of U and V

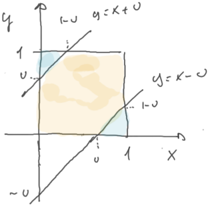
(b) Find the support of U, V

(c) Find the joint density of V, W where $W = U + V$ (not done in class)

(d) Find the joint density of U, V (not done in class)

Note that $0 < U < 1$ and for $u \in (0, 1)$

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(-u < X - Y < u) \\ &= P(u > Y - X > -u) = P(u + X > Y > X - u) \\ &= P(X - u < Y < X + u) = P((X, Y) \in \text{yellow area in the figure below}) \\ &= 1 - P((X, Y) \in \text{blue area}) = 1 - (1 - u)^2 \end{aligned}$$



In fact, on the square the density is $f_{XY} = 1$ and the blue area is two times $(1 - u)(1 - u)/2$. Hence

$$f_U(u) = \begin{cases} 2(1 - u) & u \in (0, 1) \\ 0 & \textit{otherwise} \end{cases}$$

Note that also $0 < V < 1$ and for $v \in (0,1)$

$$\begin{aligned} F_V(u) &= P(V \leq v) = 1 - P(V > v) = 1 - P(X > v, Y > v) \\ &= 1 - P(X > v)P(Y > v) = 1 - (1-v)^2 \end{aligned}$$

Hence

$$f_V(v) = \begin{cases} 2(1-v) & v \in (0,1) \\ 0 & \textit{otherwise} \end{cases}$$

To find the support of (U, V) note that $U = \max\{X, Y\} - \min\{X, Y\}$.
In fact

$$U = |X - Y| = \begin{cases} X - Y & \textit{if } X > Y \\ Y - X & \textit{if } Y > X \end{cases} = \begin{cases} \max\{X, Y\} - \min\{X, Y\} & \textit{if } X > Y \\ \max\{X, Y\} - \min\{X, Y\} & \textit{if } Y > X \end{cases}$$

Hence $U = \max\{X, Y\} - V$ and since $\max\{X, Y\} > \min\{X, Y\} = V$, when $V = v$ $U \in (0, 1 - v)$

To find the distribution of V, W note that $W = U + V = \max\{X, Y\}$. Then $V < W$ and for $v < w$ note that

$$
\begin{aligned}
F_{V\,W}(v, w) &= P(V \le v, W \le w) = P(W \le w) - P(V > v, W \le w) \\
&= P(W \le w) - P(v < V, W \le w) \\
&= P(W \le w) - P(v < X \le w)^2 = w^2 - (w - v)^2
\end{aligned}
$$

The joint density is then

$$
f_{V\,W}(v, w) = \begin{cases} 2 & 0 < v < w < 1 \\ 0 & \textit{otherwise} \end{cases}
$$

That is the joint distribution of minimum and maximum is uniform on the triangle (0,0) (1,1) (0,1) on the plane v, w

Note that we can consider U, V as a transoformation of V, W where $U = W - V$. The absolute value of the Jacobian, $|J| = 1$.

The joint density of U, V is positive on the set $\{u, v : 0 < u < 1 - v, 0 < v < 1\}$, that is

$$f_{U\,V}(v, w) = \begin{cases} 2 & 0 < u < 1 - v, 0 < v < 1 \\ 0 & otherwise \end{cases}$$

Hence, U, V is uniform on the triangle (0,0) (1,0) (0,1) on the plane u, v

Exercise. Written exam February 2021 A small museum has only three rooms A, B, C but has a painting by Monet and one by Picasso. The paintings of each room can change every week. The curator of the museum first assigns the room to Monet with equal probabilities for the three rooms and regardless of which room he had been placed in the last week. The Picasso's painting, on the other hand, if it is in the rooms A and B with probability $1/2$, certainly does not change position otherwise it goes to the room where Monet was placed. Also, if the Picasso was in the C room, the next week he would definitely be back in A. Let X_n be the Picasso room during the nth week and Y_n the position of Monet.

(a) Find the transition matrix of the chain X_n

(b) Calculate $P(X_2 = A|X_0 = A)$ (not done in class)

(c) Find the invariant distribution (not done in class)

(d) What is the approximate probability that after 100 weeks the Monet and the Picasso can be in the same room (not done in class)

Suppose the Picasso's painting is in rooom A. With probability $1/2$ it does not move and with probability $1/2$ it may move. If it moves, it follows the Monet but if he Monet is placed in room A at the end the Picasso will remain in room A. Then the probability for the Picasso's painting to remain in A is $1/2 + 1/2 \cdot 1/3 = 2/3$. The probability to go in room B is $1/2 \cdot 1/3 = 1/6$ and the probability to go in room C is $1/2 \cdot 1/3 = 1/6$. In the same way we can find the transition probabilities from the room B. The he transition matrix is then

$$P = \begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 2/3 & 1/6 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P(X_2 = A | X_0 = A) = \frac{2}{3}\frac{2}{3} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}1 = \frac{4}{9} + \frac{1}{36} + \frac{1}{6} = \frac{16+1+6}{36} = \frac{23}{36}$$

The invariant distribution satisfies $\pi P = \pi$ where $\pi = (\pi_A, \pi_B, \pi_C)$ and $\pi_A + \pi_B + \pi_C = 1$, By taking the second and third column of P we can find π solving the system

$$\begin{cases} \frac{1}{6}\pi_A + \frac{2}{3}\pi_B = \pi_B \\ \frac{1}{6}\pi_A + \frac{1}{6}\pi_B = \pi_C \\ \pi_A + \pi_B + \pi_C = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \pi_A + 4\pi_B = 6\pi_B \\ \pi_A + \pi_B = 6\pi_C \\ \pi_A + \pi_B + \pi_C = 1 \end{cases}$$

From the first equation we have $\pi_A = 2\pi_B$. By substituting π_A in the second equation we have also that $\pi_C = \frac{1}{2}\pi_B$. The third equation is then $4\pi_B + 2\pi_B + \pi_B = 2$ so that $\pi_B = \frac{2}{7}$. Hence the invariant distribution is

$$\pi = \left(\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\right)$$

For the point (d) suppose that Picasso at time $n-1$ was in room A. If Picasso does certainly not move, than we need the Monet in room A. If Picasso's painting may move than it will stay certainly with the Monet. That is

$$P(X_n = Y_n | X_{n-1} = A) = \frac{1}{2}\frac{1}{3} + \frac{1}{2} = \frac{2}{3}$$

Similarly

$$P(X_n = Y_n | X_{n-1} = B) = \frac{1}{2}\frac{1}{3} + \frac{1}{2} = \frac{2}{3} \quad \textit{while} \quad P(X_n = Y_n | X_{n-1} = C) = \frac{1}{3}$$

By taking for X_n the invariant distribution (the chain is ergodic and aperiodic) we have

$$\begin{aligned} P(X_n = Y_n) &= \sum_{s=1}^{3} P(X_n = Y_n, X_{n-1} = s) = \sum_{s=A}^{C} P(X_n = Y_n | X_{n-1} = s) P(X_{n-1} = s) \\ &\approx (4/7)(2/3) + (2/7)(2/3) + (1/7)(1/3) = 13/21 \end{aligned}$$

Exercise. Written exam June 2021 You have 2 urns, A and B. In the urn A there are 3 white balls, in the urn B there are 2 black balls. Let X_0 be the number of white balls in urn B, that is $X_0 = 0$. Suppose that you modify the urn composition in the following way. In each trial, you first take a ball from urn B and put the ball in the urn A, then you take a ball from urn A and you put the ball in urn B. Let X_1 be the number of white balls in urn B after the first complete switching and X_n be the number of white balls in urn B after n trials.

(a) Write the distribution of X_1

(b) Write the transition matrix P of the Markov Chain X_n

(c) Find the invariant distribution of the chain

Note that at each time n the number of balls in urn B is always 2.

$$P(X_1 = 0) = \frac{1}{4}$$

In fact in the first trial, with probability 1 we put a black ball from B to A. Then in A we have 3 white and 1 black balls. To have 0 white balls in B at the end of the trial we need to select the black ball. This event occurs with probability $1/4$. Since $P(X_2 = 2) = 0$ we have also

$$P(X_1 = 1) = \frac{3}{4}$$

That is $X_1 \in \{0, 1\}$ with $P(X_1 = 0) = 1/4$ and $P(X_1 = 1) = 3/4$

In general for this chain, $P(X_n = 0|X_{n-1} = 0) = 1/4$, $P(X_n = 1|X_n = 0) = 3/4$ and $P(X_n = 2|X_n = 0) = 0$.

Now suppose that $X_n = 1$, to have $X_{n+1} = 0$ we need to select the white ball in the first part of the trial and a black in the second. Then

$$P(X_{n+1} = 0|X_n = 1) = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$$

To have $X_{n+1} = 1$ we can select the white ball from B to put in A and again a white ball in A to put in B, or a black ball in B to put in A and again a black from A to put in B. That is

$$P(X_{n+1} = 1|X_n = 1) = \frac{1}{2}\frac{3}{4} + \frac{1}{2}\frac{1}{2} = \frac{5}{8}$$

To have $X_{n+1} = 2$ we need to select the black ball in the first part of the trial and a white in the second. Hence, as expected,

$$P(X_{n+1} = 1|X_n = 1) = \frac{1}{2}\frac{1}{2} = \frac{2}{8}$$

Suppose that $X_n = 2$. Then we have 2 white balls in B. hence with probability 1 we take a white and A will have 2 white and 2 black balls. Hence

$$P(X_{n+1} = 1|X_n = 2) = P(X_{n+1} = 2|X_n = 2) = \frac{1}{2}$$

In class I wrote erroneously 3/4 and 1/4!!

The transition matrix is then

$$P = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The invariant distribution satisfies the system

$$\begin{cases} \frac{1}{4}\pi_A + \frac{1}{8}\pi_B = \pi_A \\ \frac{2}{8}\pi_B + \frac{1}{2}\pi_C = \pi_C \\ \pi_A + \pi_B + \pi_C = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} 2\pi_A + \pi_B = 8\pi_A \\ 2\pi_B + 4\pi_C = 8\pi_C \\ \pi_A + \pi_B + \pi_C = 1 \end{cases}$$

From the first equation we have $\pi_B = 6\pi_A$. From the second we have $\pi_C = \frac{1}{2}\pi_B = 3\pi_A$. The third equation is then $\pi_A + 6\pi_B + 3\pi_A = 1$ so that $\pi_A = \frac{1}{10}$. Hence the invariant distribution is

$$\pi = \left(\frac{1}{10}, \frac{6}{10}, \frac{3}{10}\right)$$

This is also the limit distribution of X_n since the chain is ergodic and aperiodic.