Probability and stochastic processes

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Name and surname:

Solve 2 exercises: time 2:30 hours.

1. Alessandro and Barbara play darts. The probability that Alessandro hits the center is 1/3, while for Barbara this probability is 1/4. The two friends shoot an arrow each, Alessandro goes first and the game will end when one of the two will have hit the mark. We assume that all launches are independent.

   (a) Find the probability that Alessandro wins the game at his second shot

   Let A_i be the event "Alessandro wins the game at the ith shot" and B_i be the event "Barbara wins the game at the ith shot"

$$P(A_2) = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{6}$$

   (b) Suppose that the match ends at the first trial, what is the probability that Barabara wins

$$P(A_1) = \frac{1}{3} \quad P(B_1) = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6} \quad P(A_1 \cup B_1) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$$P(B_1 | A_1 \cup B_1) = \frac{P(B_1)}{P(A_1 \cup B_1)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

   (c) Find the probability that Barbara wins the game

$$P(B_k) = \left(\frac{2}{3} \cdot \frac{3}{4}\right)^{k-1} \frac{2}{3}\frac{1}{4} = \left(\frac{1}{2}\right)^{k-1} \frac{1}{6} \quad k = 1, 2, \ldots$$

$$P(B) = \sum_{k=1}^{\infty} P(B_k) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} \frac{1}{6} = \frac{1}{6}\frac{1}{1 - 1/2} = \frac{1}{3}$$

   (d) Find the probability that Barbara wins at the first shot given that she wins the game

$$P(B_1 | B) = \frac{P(B_1 \cap B)}{P(B)} = \frac{P(B_1)}{P(B)} = \frac{1/6}{1/3} = \frac{1}{2}$$

(e) Let X be the random variable indicating the total number of shots done by Alessandro and Barbara during the game. Find the expected value of X

Let T be the random variable indicating the number of trials until we have a winner

$$X = \begin{cases} 2T-1 & A \ \ wins \\ 2T & B \ wins \end{cases}$$

Note that $P(A) = 2/3$ and

$$P(T=k|A) = \frac{(1/2)^{k-1} 1/3}{2/3} = (1/2)(1/2)^{k-1}$$

hence $T|A \sim Geom(1/2)$.

Similarly,

$$P(T=k|B) = \frac{(1/2)^{k-1} 1/6}{1/3} = (1/2)(1/2)^{k-1}$$

hence $T|B \sim Geom(1/2)$.

$$\begin{aligned} E(X) &= E(X|A)P(A) + E(X|B)P(B) = E(2T-1|A)P(A) + E(2T|B)P(B) \\ &= (2\frac{1}{1/2} - 1)(2/3) + 2\frac{1}{1/2}(1/3) = 2 + \frac{4}{3} = 3.3333 \end{aligned}$$

A check in R

```
X=function(n){
  x=matrix(nrow=n,ncol=2)
  for (i in 1:n){
    g=0
    gioco=TRUE
    while(gioco){ provaA=rbinom(1,1,1/3)
      g=g+1
      if (provaA==1) {gioco=FALSE
                      vince="A"}
        else        {g=g+1
          provaB=rbinom(1,1,1/4)
        if (provaB==1) {vince="B"
                        gioco=FALSE}}
    }
  x[i,]=c(g,vince)}
return(x)}
n=100000
ris=X(n=n)
print(mean(as.numeric(ris[,1])))

[1] 3.33372
```

2. Let (X, Y) be a random variable with density

$$f(x,y) = \begin{cases} c\,x & 0 < x < 1, 0 < y < x^2 \\ 0 & otherwise \end{cases}$$

a) Find c and the marginal densities of X and Y

b) Are X and Y independent

c) Find the covariance between X and Y

d) Let $Z = X + Y$ and $V = X - Y$, find the joint density of Z, V. (Write the density and the region where the density is strictly positive)

$$1 = c\int_0^1 \left(\int_0^{x^2} x dy\right) dx = c\int_0^1 x^3 dx = \frac{c}{4}$$

Hence $c = 4$

$$f_x(x) = \int_0^{x^2} 4x dy = 4x^3 \quad x \in (0,1)$$

$$f_Y(y) = \int_{\sqrt{y}}^1 4x dx = \left[2x^2\right]_{\sqrt{y}}^1 = 2(1-y) \quad y \in (0,1)$$

$$E(X) = \int_0^1 x\, 4x^3 dx = \frac{4}{5} \quad E(Y) = \int_0^1 y2(1-y)dy = \frac{1}{3}$$

$$E(XY) = \int_0^1 \left(\int_0^{x^2} xy\, 4x\, dy\right) dx = \int_0^1 4x^2 \frac{1}{2} x^4 dx = \frac{2}{7}$$

$$Cov(X,Y) = \frac{2}{7} - \frac{4}{15} = \frac{30-28}{105} = \frac{2}{105}$$

Note that $X \sim Beta(4,1)$ and $Y|X = x \sim Unif(0, x^2)$

Let A be the set where the density of Z, V is strictly positive.

$$f_{ZV}(z,v) = z + v \quad (z,v) \in A$$

In fact $X = \frac{1}{2}(Z+V)$ and $Y = \frac{1}{2}(Z-V)$. Then

$$J = det\begin{pmatrix} \frac{\partial x(z,v)}{\partial z} & \frac{\partial x(z,v)}{\partial v} \\ \frac{\partial y(z,v)}{\partial z} & \frac{\partial y(z,v)}{\partial v} \end{pmatrix} = det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}$$

hence

$$f_{ZV}(z,v) = f_{XY}\left(x = \frac{z+v}{2}, y = \frac{z-y}{2}\right)|J| = 4\frac{z+v}{2}\frac{1}{2} = z + v \quad (z,v) \in A$$

To find the set A note that

$$0 < \frac{1}{2}(z+v) < 1 \Rightarrow \begin{cases} v > -z \\ v < 2-z \end{cases}$$

$$0 < \frac{1}{2}(z-v) < \frac{1}{4}(z+v)^2 \Rightarrow \begin{cases} v < z \\ v^2 + v(2z+2) + z^2 - 2z > 0 \end{cases}$$

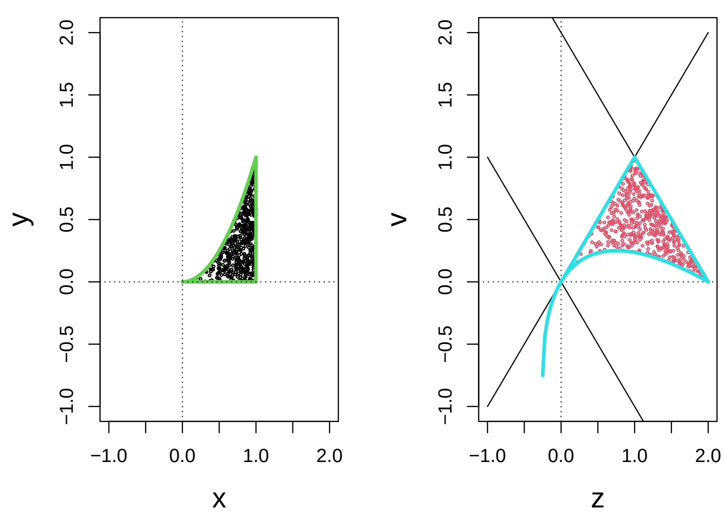
By considering the contraints $v > -z, v < z$ we see that $z > 0$.

Note that $f(v) = v^2 + v(2z+2) + z^2 - 2z = 0 \Rightarrow v = -(z+1) \pm \sqrt{1+4z}$. Then the second degree equation $f(v) = 0$ has two real solutions $\forall z > 0$. Moreover the stationary point for which $f'(v) = 0$ (i.e. $v = -(z+1)$) is a minimum since $f''(v) > 0$. Hence $\forall\, z > 0$ the second degree inequality is satisfied when

$$v < -(z+1) - \sqrt{1+4z} \quad or \quad v > -(z+1) + \sqrt{1+4z}.$$

Finally note that since $v > -z$ then $v > -z-1-\sqrt{1+4z}$ then the first possibility above can be excluded. In addition since for $z > 0$, $-z-1+\sqrt{1+4z} > -z$ the region where the density of (Z, V) is strictly positive is given by the set

$$\begin{cases} v < 2-z \\ v < z \\ v > -(z+1) + \sqrt{1+4z} \end{cases}$$



3. The joint density of (X, Y) is

$$f(x,y) = \begin{cases} k(1-\sqrt{y/x}) & 0<x<1, 0<y<x \\ 0 & otherwise \end{cases}$$

(a) Find k and the marginal density of X and Y. Are X and Y independent?

(b) Find the density of (V, Z) where $V = X$ and $Z = \sqrt{Y/X}$. (Write the density and the region where the density is strictly positive) Are V and Z independent?

(c) Find the density of $max\{V, Z\}$

(d) Find $P(V^2 + Z^2 > 1/2)$

$k = 6$

$$f_X(x) = \int_0^x 6(1-\sqrt{y/x})dy = 6\left(x - \frac{1}{x^{1/2}}\frac{2}{3}x^{3/2}\right) = 6\left(x - \frac{2}{3}x\right) = 2x \quad x \in (0,1)$$

$X \sim Beta(2, 1)$

$$f_Y(y) = \int_y^1 6(1-\sqrt{y/x})dx = 6\left(1 - y - 2\sqrt{y}(1-\sqrt{y})\right) = 6(1+y-2\sqrt{y}) \quad y \in (0,1)$$

X and Y are dependent.

$X = V$, $Z^2 = Y/V$, then $Y = V Z^2$

$$J = det\begin{pmatrix} \frac{\partial x(v,z)}{\partial v} & \frac{\partial x(v,z)}{\partial v} \\ \frac{\partial y(v,z)}{\partial v} & \frac{\partial y(v,z)}{\partial z} \end{pmatrix} = det\begin{pmatrix} 1 & 0 \\ z^2 & 2vz \end{pmatrix} = 2vz$$

$$f_{VZ}(v,z) = 6\left(1 - \sqrt{\frac{vz^2}{v}}\right)2vz = 6(1-z)z\,2v \quad 0<v<1 \; 0<z<1$$

V and Z are independent. Moreover $F_V(v) = v^2$ for $v \in (0,1)$. $F_Z(z) = 3z^2 - 2z^3$ for $z \in (0,1)$

$$P(max\{V,Z\} < u) = u^2(3u^2 - 2u^3) = 3u^4 - 2u^5$$

and the density of the maximum is $12u^3 - 10u^4$

Set $S = V^2$ and $T = Z^2$

$$f_S(s) = f_V(v = \sqrt{s})\frac{d}{ds}\sqrt{s} = 2\sqrt{s}\frac{1}{2\sqrt{s}} = 1 \quad s \in (0,1)$$

$$f_T(s) = f_Z(z = \sqrt{t})\frac{d}{dt}\sqrt{t} = 6(1-\sqrt{t})\sqrt{t}\frac{1}{2\sqrt{t}} = 3(1-\sqrt{t}) \quad s \in (0,1)$$

$$\begin{aligned} P(V^2+Z^2<1/2) &= P(S+T<1/2) = \int_0^{1/2}\left(\int_0^{1/2-s} 3(1-\sqrt{t})dt\right)ds \\ &= 3\int_0^{1/2} \frac{1}{2} - s - \frac{2}{3}(\frac{1}{2}-s)^{3/2}ds = \cdots = \frac{3}{8} - \frac{4}{5}\frac{1}{2^{5/2}} \end{aligned}$$

4. A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes then it is equally likely to be either of the other two possibilities. Let $\{X_n, n \geq 0\}$ be the Markov chain indicating the weather condition on day n. Suppose that at time 0 the three weather states have the same probability

   a) Write the transition matrix of the Markov chain
   b) Find $P(X_0 = sunny | X_2 = sunny)$
   c) Find the invariant distribution of the chain
   d) In the long run, what is the probability to have three rainy days in a row
   e) In a different town we may have two sunny days but we cannot have cloudy days. In this town the probability to have a rainy day is 0.3 if the last two days were sunny. Moreover if in the last two days, at least one was a rainy day the probability to have a rainy day is 0.8. Let $T_n = (X_n, X_{n-1})$ be the random variable indicating the weather conditions of two consecutive days. Write the transition matrix for T_n

The three states are $\{sunny, cloudy, rainy\}$

$$P = \begin{array}{c} S \\ C \\ R \end{array} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

Note that

$$P(X_2 = sunny | X_0 = sunny) = p_{SS}(2) = 0 \cdot 0 + (1/2) \cdot (1/4) + (1/2) \cdot (1/4) = 1/4$$

$$P(X_0 = sunny) = 1/3$$

$$P(X_2 = sunny) = (1/3)p_{SS}(2) + (1/3)p_{CS}(2) + (1/3)p_{RS}(2) = (1/3)(1/4+3/16+3/16) = \frac{10}{48}$$

$$\begin{aligned} P(X_0 = sunny | X_2 = sunny) &= \frac{P(X_0 = sunny \cap X_2 = sunny)}{P(X_2 = sunny)} \\ &= \frac{P(X_2 = sunny | X_0 = sunny) P(X_0 = sunny)}{P(X_2 = sunny)} = \frac{(1/4)(1/3)}{10/48} = 4/10 \end{aligned}$$

The invariant distribution is $\pi = (1/5, 2/5, 2/5)$

The probability to have three rainy days in a row is

$$\pi_R p_{RR}(1) p_{RR}(1) = (2/5)(1/2)(1/2) = 1/10$$

The new Markov chain has the following states

$$\{SS, SR, RS, RR\}$$

$$P = \begin{array}{c} SS \\ SR \\ RS \\ RR \end{array} \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.8 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}$$

In fact

$$\begin{aligned} P(T_{n+1} = SS | T_n = SS) &= P(X_{n+1} = S, X_n = S | X_n = S, X_{n-1} = S) \\ &= P(X_{n+1} = S | X_n = S, X_{n-1} = S) P(X_n = S | X_n = S, X_{n-1} = S) \\ &= 0.7 \cdot 1 \end{aligned}$$

$$\begin{aligned} P(T_{n+1} = RS | T_n = SS) &= P(X_{n+1} = R, X_n = S | X_n = S, X_{n-1} = S) \\ &= P(X_{n+1} = R | X_n = S, X_{n-1} = S) P(X_n = S | X_n = S, X_{n-1} = S) \\ &= 0.3 \cdot 1 \end{aligned}$$

$$\begin{aligned} P(T_{n+1} = SR | T_n = RS) &= P(X_{n+1} = S, X_n = R | X_n = R, X_{n-1} = S) \\ &= P(X_{n+1} = S | X_n = R, X_{n-1} = S) P(X_n = R | X_n = S, X_{n-1} = S) \\ &= 0.2 \cdot 1 \end{aligned}$$

And so on...