Stochastic Processes

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Surname and name _____

Identification number _____

Time: 2 hours to solve at least 2 exercises

1. You distribute n balls into 3 empty boxes U_1, U_2, U_3 at random and independently. (Each box has the same probability to to receive a ball and multiple occupancy being permitted)

   Find the probability that

   A. the box U_1 is empty;

   B. the boxes U_1 and U_2 are empty;

   C. two boxes are empty

   D. only the first box is empty

   E. at least one of the boxes is empty

   F. the mean number of balls of the first box U_1.

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2. You choose a point (X, Y) where X is Uniform(0,1) and the density of $Y|X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} ky & if\ y \in (0, x) \\ 0 & otherwise \end{cases}$$

   A. Find the marginal density of Y

   B. Find the covariance of (X, Y)

   C. Find the distribution of $Z = X + Y$ or alternatively that of $W = Y/X$

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3. In the $[0, 1]$ interval you choose independently n points uniformly distributed. Let $X_1, X_2, \ldots, X_n$ be the random variables indicating these points. Find the distribution (or if you are not able find only the expected value) of the the following random variable

   A. $Y = \min(X_1, X_2, \ldots, X_n)$

   B. $W = \max(X_1, X_2, \ldots, X_n)$

   C. $Z = W - Y$

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4. You have separate parents who live in two opposite parts of the city, one in the north (N) and one in the south (S). During the weekends, your visit strategy is as follows. If you stayed at home during the last weekend (C), you go to N with probability p and to S with probability $q = 1 - p$. If, on the other hand, the previous weekend you went to N or S, you decide to stay at home with probability $1 - r$ or to go " to the other parent " with probability r.

   Let $\{X_n, n \geq 1\}$ be the process that describes your choice over the various weekends.

   A. Write the transition matrix of the process

   B. Determine if the chain is irreducible and periodic or aperiodic

   C. Find the invariant distribution

   D. Suppose that $r = 0$. In this case the chain is periodic or aperiodic? Find the distribution of the random variable T given by the time of the first week-end at home given that at time 0 you were at home