Stochastic Processes

Prof. Andrea Tancredi

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1. You distribute n balls into 3 empty boxes U_1, U_2, U_3 at random and independently. (Each box has the same probability to to receive a ball and multiple occupancy being permitted)

   Find the probability that

   A. the box U_1 is empty;

   B. the boxes U_1 and U_2 are empty;

   C. two boxes are empty

   D. only the first box is empty

   E. at least one of the boxes is empty

   F. the mean number of balls of the first box U_1.

   A. Let E_i be the event "The i-th box is empty". We have 3^n arrangements for the n balls and 2^n arrangements with U_1 empty. Then

   $$P(E_1) = 2^n/3^n$$

   B. In this case all the balls must go in U_3 and we have just 1 arrangement of the n balls.

   $$P(E_1 \cap E_2) = P(U_3 \text{ is full}) = 1/3^n$$

   C. We have three possibilities to select the full box. Hence

   $$P(\text{ two boxes will remain empty}) = 3/3^n$$

   D. There are 2^n arrangements where U_1 is empty. From these 2^n outcomes we need to exclude the arrangement where U_2 is also empty and the arrangement where U_3 is empty. Hence we have $2^n - 2$ aragements where only U_1 is empty and

   $$P(\text{only the first box is empty}) = \frac{2^n - 2}{3^n}$$

   E.

   $$\begin{aligned} P(\text{at least one is empty}) &= P(E_1 \cup E_2 \cup E_3) \\ &= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) \\ &- P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) \\ &= 3P(E_1) - 3P(E_1 \cap E_2) + 0 \\ &= 3\left(\frac{2}{3}\right)^n - 3\left(\frac{1}{3}\right)^n \end{aligned}$$

   F. This is the mean of a Binomial (n,1/3) which is $n/3$

2. You choose a point (X, Y) where X is Uniform(0,1) and the density of $Y|X = x$ is

$$f_{Y|X}(y|z) = \begin{cases} ky & if\ y \in (0,x) \\ 0 & otherwise \end{cases}$$

A. Find the marginal density of Y

B. Find the covariance of (X, Y)

C. Find the distribution of $Z = X + Y$ or alternatively that of $W = Y/X$

A.

$$1 = \int_0^x ky\,dy = k\,\frac{y^2}{2}\bigg|_1^x = \frac{k}{2}x^2$$

Hence $k = 2/x^2$,

$$f_{XY}(x,y) = \begin{cases} 2\,y/x^2 & 0 < y < x < 1 \\ 0 & otherwise \end{cases}$$

and

$$f_Y(y) = \int_y^1 \frac{2\,y}{x^2}\,dx = -\,2\,y\,\frac{1}{x}\bigg|_y^1 = 2(1-y) \quad y \in (0,1)$$

B.

$$E(Y|X=x) = \int y f_{Y|X}(y|x)dy = \int_0^x y\frac{2}{x^2}y\,dy = \frac{2x}{3}$$

$$E(Y) = E(E(Y|X)) = E(2X/3)) = \frac{2}{3}E(X) = \frac{2}{3}\frac{1}{2} = \frac{1}{3}$$

$$E(XY) = E(E(XY|X)) = E(X(2X/3)) = (2/3)E(X^2) = \frac{2}{3}\frac{1}{3} = \frac{2}{9}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{2}{9} - \frac{1}{2}\frac{1}{3} = \frac{1}{18}$$

C. Consider W. Since $Y < X$, the image of W is (0,1). For $w \in (0,1)$

$$\begin{aligned} F_W(w) &= P(W \le w) = P(Y/X \le w) = P(Y \le wX) \\ &= \int_0^1 \left[\int_0^{wx} f_{XY}(x,y)dy\right] dx = \int_0^1 \frac{2}{x^2}\left[\int_0^{wx} y\,dy\right] dx \\ &= \int_0^1 \frac{2}{x^2}\,\frac{w^2x^2}{2}dx = w^2 \end{aligned}$$

The density of W is then

$$f_W(w) = 2w \quad w \in (0,1)$$

3. In the $[0,1]$ interval you choose independently n points uniformly distributed. Let $X_1, X_2, \ldots, X_n$ be the random variables indicating these points. Find the distribution (or if you are not able find only the expected value) of the the following random variable

   A. $Y = \min(X_1, X_2, \ldots, X_n)$

   B. $W = \max(X_1, X_2, \ldots, X_n)$

   C. $Z = W - Y$

A. The image of Y is $[0,1]$ For $y \in (0,1)$,

$$F_Y(y) = P(Y \leq y) = 1 - P(Y > y) = 1 - P\left(\bigcap_{j=1}^{n}(X_j > y)\right) = 1 - \prod_{j=1}^{n} P(X_j > y) = 1 - (1-y)^n$$

Moreover $F_Y(y) = 0$, if $y \leq 0$ and $F_Y(y) = 1$ if $y \geq 1$.
The density of Y is

$$f_Y(y) = n(1-y)^{n-1}, \quad 0 < y < 1.$$

B. The image of $W$is $[0,1]$. For $w \in (0,1)$,

$$F_W(w) = P(W \leq w) = P\left(\bigcap_{j=1}^{n}(X_j \leq w)\right) = \prod_{j=1}^{n} P(X_j \leq w) = w^n$$

Moroever $F_W(w) = 0$ if $w \leq 0$ and $F_Y(w) = 1$ if $w \geq 1$. The density of W is

$$f_W(w) = nw^{n-1}, \quad 0 < w < 1.$$

C. Since $P(W \leq w) = P(Y \leq y, W \leq w) + P(Y > y, W \leq w)$ The joint distribution function of (Y, W) in the set $\{(y,w) : 0 < y < w < 1\}$ is

$$\begin{aligned} F_{Y,W}(y,w) &= P(Y \leq y, W \leq w) = P(W \leq w) - P(Y > y, W \leq w) \\ &= w^n - P\left(\bigcap_{j=1}^{n}(y < X_j < w)\right) = w^n - (w-y)^n \end{aligned}$$

The joint density is

$$f_{YW}(y,w) = \frac{\partial^2}{\partial w \partial y} F_{Y,W}(y,w) = n(n-1)(w-y)^{n-2}, \quad 0 < w < y < 1$$

To find the density of of $Z = W - Y$ we consider $Z = Y - W$ and $S = W$. The inverse transformation is $W = Z + S$ and $Y = S$ and the Jacobian is 1. Since $0 < y < w < 1$, the density of (Z, S) is defined on the set $0 < s < z + s < 1$, which is $0 < s < 1$ and $0 < z < 1 - s$. The density is

$$f_{S,Z}(s,z) = f_{Y,W}(s, z+s) = n(n-1)z^{n-2}, \quad 0 < s < 1; z < 1 - s,$$

The marginal density of Z is

$$f_Z(z) = \int_0^{1-z} n(n-1)z^{n-2} ds = n(n-1)z^{n-2}(1-z), \quad 0 < z < 1.$$

The expected values of Y and W are

$$EY = \frac{1}{n+1}; EW = \frac{n}{n+1};$$

In fact

$$\begin{aligned} E(Y) &= \int_0^1 y f_y(y)\, dy = \int_0^1 yn(1-y)^{n-1}\, dy \\ &= -y(1-y)^2\big|_0^1 + \int_0^1 (1-y)^n dy = 0 - \frac{1}{n+1}(1-y)^{n+1}\Big|_0^1 = \frac{1}{n+1} \end{aligned}$$

$$E(W) = \int_0^1 w f_w(w)\, dw = \int_0^1 nw^n\, dw = \frac{n}{n+1}\, w^{n+1}\big|_0^1 = \frac{n}{n+1}$$

and

$$EZ = EW - EY = \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}$$

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4. You have separate parents who live in two opposite parts of the city, one in the north (N) and one in the south (S). During the weekends, your visit strategy is as follows. If you stayed at home during the last weekend (C), you go to N with probability p and to S with probability $q = 1 - p$. If, on the other hand, the previous weekend you went to N or S, you decide to stay at home with probability $1 - r$ or to go " to the other parent " with probability r.

   Let $\{X_n, n \geq 1\}$ be the process that describes your choice over the various weekends.

   A. Write the transition matrix of the process
   B. Determine if the chain is irreducible and periodic or aperiodic
   C. Find the invariant distribution
   D. Suppose that $r = 0$. In this case the chain is periodic or aperiodic? Find the distribution of the random variable T given by the time of the first week-end at home given that at time 0 you were at home

A. Let (C, N, S) be the states of the chain. The transition matrix is

$$P = \begin{pmatrix} 0 & p & 1-p \\ 1-r & 0 & r \\ 1-r & r & 0 \end{pmatrix}$$

B. The chain is irreducible. In fact all the states communicate. It is aperiodic since $p_{CC}(n) > 0$ for $n = 2, 3, ..$ and the $g.c.d.$ is 1

C. The invariant distribution satisfies the following system

$$\begin{cases} \pi_C\, 0 + \pi_N\,(1-r) + \pi_S\,(1-r) & = & \pi_C \\ \pi_C\, p + \pi_N\, 0 + \pi_S\, r & = & \pi_N \\ \pi_C + \pi_N + \pi_S & = & 1 \end{cases}$$

By summing the first two equations and subtracting the third one we obtain

$$\pi_N = \frac{1}{r+1} + \frac{p-2}{r+1}\pi_C$$

Substituting π_N into the second equation we have

$$\pi_S = \frac{1}{r(r+1)} - \frac{2+rp}{r(r+1)}\pi_C$$

Substituting π_N and π_S in the last equation we find

$$\pi_C = \frac{r^2-1}{r^2-r-2} = \frac{1-r}{2-r}$$

Moreover

$$\pi_N = \frac{1}{r+1} + \frac{(1-r)(p-2)}{(r+1)(2-r)} = \frac{r}{r+1} + \frac{(1-r)(p-r)}{(r+1)(2-r)}$$

and

$$\pi_S = \frac{1}{r(r+1)} - \frac{(2+rp)}{r(r+1)}\frac{(1-r)}{(2-r)} = \frac{1}{1+r} - \frac{1+p-rp+r}{(r+1)(2-r)}$$

D. When $r = 0$ the transition matrix is

$$P = \begin{pmatrix} 0 & p & 1-p \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $p_{CC}(n) > 0$ for $n = 2, 4, 6, ...$ Then the period of the chain is 2. Moreover if T is the the time of the first return $P(T = 2|X_0 = C) = 1$, that is $T|X_0 = C$ is concentrated on 2.