Stochastic Processes

Prof. Andrea Tancredi

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Surname and name ________________________________

Identification number ________________________________

Time: 2 hours to solve at least 2 exercises

1. A box contains c colored balls each of a different color. Consider a sequence of independent draws with replacement from the box.

   A. Calcolare la probabilità che le prime due palline estratte abbiano lo stesso colore

   B. Let Y_1 be the r.v indicating the number of extractions necessary to observe a ball of a different color from that observed in the first extraction. (Exclude the first draw from the count). What is the distribution of Y_1

   C. Let Y_j be the r.v indicating the number of extractions required to observe a new color, after j have been observed. What is the distribution of Y_j?

   D. Alternatively to F. Let T be the number of extractions necessary to observe all colors. If $c = 4$ what is the expected value of T.

   E. Calculate the probability that in n extractions a given color is never observed.

   F. Alternatively to D. Calculate the expected value for number of colors observed in n extractions

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2. Let (U, V) be a unform distribution on the square $(0, 0), (0, 1)$ $(1, 1)(1, 0)$

   A. Calculate $P(V^2 > U)$

   B. Find the density of $Z = U + V^2$

   C. Calculate $P(V^2 > U|U + V < 1)$

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3. Let X and Y two random variable with joint density

$$f(x,y) = \begin{cases} \frac{1}{2} + 2xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   A. Find the marginal density of X

   B. Have X and Y the same marginal density?

   C. Are X and Y independent?

   D. Calculate $P(X + Y < 1)$.

E. Calculate the covariance between X and Y.

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4. Suppose that five components are functioning simultaneously, that the lifetimes of the components are i.i.d., and that each lifetime has the exponential distribution with parameter β Let T_1 denote the time from the beginning of the process until one of the components fails, and let T_5 denote the total time until all five components have failed.

   A. Explicitly write down the marginal distributions of T_1 and T_5

   B. Find the covariance between T_1 and T_5

   C. Find the joint distribution of T_1 and T_5

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5. A small museum has only three rooms A, B, C but has a painting by Monet and one by Picasso. The paintings of each room can change every week. The curator of the museum first assigns the room to Monet with equal probabilities for the three rooms and regardless of which room he had been placed in the last week. The Picasso's painting, on the other hand, if it is in the rooms A and B with probability $1/2$, certainly does not change position otherwise it goes to the room where Monet was placed. Also, if the Picasso was in the C room, the next week he would definitely be back in A. Let X_n be the Picasso room during the nth week and Y_n the position of Monet.

   A. Find the transition matrix of the chain X_n

   B. Calculate $P(X_2 = A | X_0 = A)$

   C. Find the invariant distribution

   D. What is the approximate probability that after 100 weeks the Monet and the Picasso can be in the same room

   E. Determine whether, conditionally on $X_0 = A$, the random variables the X_2 and Y_1 are dependent or not.