Stochastic Processes

Prof. Andrea Tancredi

Written exam 16-02-2021

Surname and name ______________________________

Identification number ______________________________

Time: 2 hours to solve at least 2 exercises

1. A box contains c colored balls each of a different color. Consider a sequence of independent draws with replacement from the box.

   A. Calculate the probability that the first two balls have the same colour

   Let C_i be the color of the ith ball.

$$\begin{aligned} P(C_1 = C_2) &= \sum_{j=1}^{c} P(C_1 = C_2 \cap C_1 = j) = \sum_{j=1}^{c} P(C_1 = C_2 | C_1 = j) P(C_1 = j) \\ &= \sum_{j=1}^{c} P(C_2 = j | C_1 = j) P(C_1 = j) = \sum_{j=1}^{c} P(C_2 = j) P(C_1 = j) = \sum_{j=1}^{c} \frac{1}{c}\frac{1}{c} = \frac{1}{c} \end{aligned}$$

   B. Let Y_1 be the r.v indicating the number of extractions necessary to observe a ball of a different color from that observed in the first extraction. (Exclude the first draw from the count). What is the distribution of Y_1

   Note that to have $Y_1 = k$ we need to have the first $k-1$ balls after the first one equal to C_1 and then a ball different from C_1. Given the independence we have

$$P(Y_1 = k) = \left(\frac{1}{c}\right)^{k-1} \left(1 - \frac{1}{c}\right)$$

   That is $Y_1 \sim Geometric((c-1)/c)$

   C. Let Y_j be the r.v indicating the number of extractions required to observe a new color, after j have been observed. What is the distribution of Y_j?

   When j colors have been observed the probability to observe a new color is $c - j/c$. Similarly to the previous case we have

$$P(Y_j = k) = \left(\frac{j}{c}\right)^{k-1} \left(1 - \frac{j}{c}\right)$$

   That is $Y_1 \sim Geometric((c-j)/c)$

   D. Alternatively to F. Let T be the number of extractions necessary to observe all colors. If $c = 4$ what is the expected value of T.

$$T = 1 + Y_1 + Y_2 + Y_3$$

$$E(T) = 1 + \frac{1}{3/4} + \frac{1}{2/4} + \frac{1}{1/4} = 4\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right] = \frac{25}{3}$$

E. Calculate the probability that in n extractions a given color is never observed.

$$P(color\ r\ never\ observed) = (1 - 1/c)^n$$

F. Alternatively to D. Calculate the expected value for number of colors observed in n extractions

Let Z be the number of colors observed in n extractions

$$Z = \sum_{r=1}^{c} Z_r$$

where

$$Z_r = \begin{cases} 1 & if\ color\ r\ is\ observed \\ 0 & otherwise \end{cases}$$

$$E(Z_r) = \sum_{r=1}^{c} E(Z_r) = \sum_{r=1}^{c} [1 - (1 - 1/c)^n] = c[1 - (1 - 1/c)^n]$$

---

2. Let (U, V) be a unform distribution on the square $(0,0), (0,1)$ $(1,1)(1,0)$

A. Calculate $P(V^2 > U)$

$$\begin{aligned} P(V^2 > U) &= P(V > \sqrt{U}) = \int_0^1 \int_{\sqrt{u}}^1 dv\ du = \int_0^1 (1 - \sqrt{u}) du \\ &= u|_0^1 - \frac{2}{3\sqrt{u}}\bigg|_0^1 = \frac{1}{3} \end{aligned}$$

B. Find the density of $Z = U + V^2$

The density of $W = V^2$ is

$$f_w(w) = \begin{cases} \frac{1}{2\sqrt{w}} & w \in (0,1) \\ 0 & otherwise \end{cases}$$

The support of Z is $(0, 2)$. The of density of $Z = U + W$ is then

$$f_Z(z) = \int_{-\infty}^{\infty} f_U(u) f_W(z - u) du$$

When $z \in (0, 1]$

$$f_Z(z) = \int_0^z \frac{1}{2\sqrt{z - u}} du = \sqrt{z}$$

When $z \in (1, 2)$

$$f_Z(z) = \int_{z-1}^1 \frac{1}{2\sqrt{z - u}} du = 1 - \sqrt{z - 1}$$

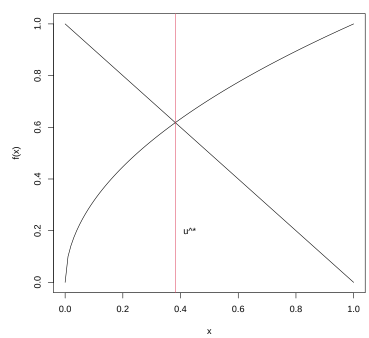
C. Calculate $P(V^2 > U | U + V < 1)$

$$P(V^2 > U|U+V<1) = \frac{P(V>\sqrt{U}, V<1-U)}{P(V<1-U)} = \frac{P(\sqrt{U}<V<1-U)}{P(V<1-U)}$$

$$P(V<1-U) = 1/2$$

$$P(\sqrt{U}<V<1-U) = \int_0^{u^*}\int_{\sqrt{u}}^{1-u} dv\ du$$

Note that the point $u^* \in (0,1)$ is such that $\sqrt{u^*} = 1-u^*$. That is, it is the solution of $u = (1-u)^2$. Hence consider $u^2 - 3u + 1 = 0$. The roots are $3/2 \pm \sqrt{9-4}/2$ hence $u^* = 3/2 - \sqrt{5}/2$



$$P(\sqrt{U}<V<1-U) = \int_0^{u^*}(1-u-\sqrt{u})\ du = \left(u - \frac{1}{2}u^2 - \frac{2}{3}u^{3/2}\right)\Big|_0^{u^*} = u^* - \frac{1}{2}u^* - \frac{2}{3}(u^*)^{3/2}$$

---

3. Let X and Y two random variable with joint density

$$f(x,y) = \begin{cases} \frac{1}{2} + 2xy & 0<x<1, 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

A. Find the marginal density of X

$$f(x) = \int_0^1\left(\frac{1}{2}+2xy\right)dy = \frac{1}{2}x\Big|_0^1 + x\ y^2\Big|_0^1 = \frac{1}{2} + x \quad xin(0,1)$$

B. Have X and Y the same marginal density?

Yes.

$$f_y(y) = \frac{1}{2} + y \quad y \in (0,1)$$

C. Are X and Y independent?

No

$$f(x,y) \neq f(x)f(y) = \frac{1}{4} + \frac{1}{2}x + \frac{1}{2}y + xy$$

D. Calculate $P(X+Y<1)$.

$$\begin{aligned} P(Y<1-X) &= \int_0^1 \int_0^{1-x} \frac{1}{2} + 2xy\,dy\,dx = \int_0^1 \frac{1}{2}(1-x) + x(1-x)^2 dx \\ &= \int_0^1 \frac{1}{2} + \frac{1}{2}x - 2x^2 + x^3 dx = \frac{1}{2} + \frac{1}{4} - \frac{2}{3} + \frac{1}{4} = \frac{1}{3} \end{aligned}$$

E. Calculate the covariance between X and Y.

$$E(X) = E(Y) = 7/12$$

$$E(XY) = \int_0^1 \int_0^1 \frac{1}{2}xy + 2x^2y^2 dydx = \int_0^1 \frac{x}{4} + \frac{2x^2}{3} dx = \frac{1}{8} + \frac{2}{9} = 25/72$$

$$Cov(X,Y) = 25/72 - (7/12)^2 = 1/144$$

---

4. Suppose that five components are functioning simultaneously, that the lifetimes of the components are i.i.d., and that each lifetime has the exponential distribution with parameter β Let T_1 denote the time from the beginning of the process until one of the components fails, and let T_5 denote the total time until all five components have failed.

A. Explicitly write down the marginal distributions of T_1 and T_5

$$\begin{aligned} F_{T_1}(t_1) &= P(T_1 \leq t_1) = 1 - P(T_1 > t_1) = 1 - P\left(\bigcap_{j=1}^{5}(X_j > t_1)\right) \\ &= 1 - \prod_{j=1}^{5} P(X_j > t_1) = 1 - e^{-5\beta t_1} \end{aligned}$$

T_1 is Exponential (5β)

$$\begin{aligned} F_{T_5}(t_5) &= P(T_5 \leq t_5) = P\left(\bigcap_{j=1}^{5}(X_j \leq t_5)\right) \\ &= \prod_{j=1}^{5} P(X_j \leq t_5) = \left(1 - e^{-\beta t_5}\right)^5 \end{aligned}$$

B. Find the joint distribution of T_1 and T_5

$$\begin{aligned} F_{T_1,T_2}(t_1,t_5) &= P(T_1 \le t_1, T_5 \le t_5) = P(T_5 \le t_5) - P(T_1 > t_1, T_5 \le t_5) \\ &= P(T_5 \le t_5) - P\left(\bigcap_{j=1}^{5}(t_1 < X_j < t_5)\right) = \left(1 - e^{-\beta t_5}\right)^5 - \left(e^{-\beta t_1} - e^{-\beta t_5}\right)^5 \end{aligned}$$

C. Find the covariance between T_1 and T_5

---

5. A small museum has only three rooms A, B, C but has a painting by Monet and one by Picasso. The paintings of each room can change every week. The curator of the museum first assigns the room to Monet with equal probabilities for the three rooms and regardless of which room he had been placed in the last week. The Picasso's painting, on the other hand, if it is in the rooms A and B with probability 1/2, certainly does not change position otherwise it goes to the room where Monet was placed. Also, if the Picasso was in the C room, the next week he would definitely be back in A. Let X_n be the Picasso room during the nth week and Y_n the position of Monet.

   A. Find the transition matrix of the chain X_n

   Suppose the Picasso's painting is in rooom A. With probability 1/2 it does not move and with probability 1/2 it may move. If it moves, it follows the Monet but if he Monet is placed in room A at the end the Picasso will remain in room A. Then the probability for the Picasso's painting to remain in A is 1/2+1/2*1/3=2/3. The probability to go in room B is 1/2*1/3=1/6 and the probability to go in room C is 1/2*1/3=1/6 . That is the transition matrix is

$$P = \begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 2/3 & 1/6 \\ 1 & 0 & 0 \end{pmatrix}$$

   B. Calculate $P(X_2 = A|X_0 = A)$

$$P(X_2 = A|X_0 = A) = \frac{2}{3}\frac{2}{3} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}1 = \frac{4}{9} + \frac{1}{36} + \frac{1}{6} = \frac{16+1+6}{36} = \frac{23}{36}$$

   C. Find the invariant distribution

   $\pi = (4/7, 2/7, 1/7)$

   It is also the limit distribution of the chain since X_n is ergodic and aperidic.

   D. What is the approximate probability that after 100 weeks the Monet and the Picasso can be in the same room

   Suppose that Picasso at time $n-1$ was in room A. If Picasso does certainly not move, than we need the Monet in room A. If Picasso's painting may move than it will stay certainly with the Monet. That is

$$P(X_n = Y_n | X_{n-1} = A) = \frac{1}{2}\frac{1}{3} + \frac{1}{2} = \frac{2}{3}$$

Similarly

$$P(X_n = Y_n | X_{n-1} = B) = \frac{1}{2}\frac{1}{3} + \frac{1}{2} = \frac{2}{3}$$

while

$$P(X_n = Y_n | X_{n-1} = C) = \frac{1}{3}$$

Hence

$$\begin{aligned} P(X_n = Y_n) &= \sum_{s=1}^{3} P(X_n = Y_n, X_{n-1} = s) = \sum_{s=A}^{C} P(X_n = Y_n | X_{n-1} = s) P(X_{n-1} = s) \\ &= (4/7) * (2/3) + (2/7) * (2/3) + (1/7) * (1/3) = 13/21 \end{aligned}$$

E. Determine whether, conditionally on $X_0 = A$, the random variables X_2 and Y_1 are dependent or not.

$$P(Y_1 = A | X_0 = A) = 1/3 \quad P(X_2 = A | X_0 = A) = 23/36$$

$$P(X_2 = A, Y_1 = A | X_0 = A) = P(X_2 = A | Y_1 = A, X_0 = A) P(Y_1 = A | X_0 = A)$$

Note that given $Y_1 = A$ and $X_0 = A$ we know that $X_1 = A$. In fact, if Picasso does not move, it will stay in A. If it will move will stay with the Monet which is in A. Hence

$$P(X_2 = A, Y_1 = A | X_0 = A) = P(X_2 = A | X_1 = A)\frac{1}{3} = \frac{2}{3}\frac{1}{3} \neq \frac{23}{36}\frac{1}{3}$$

Hence conditionally on $X_1 = A$ Y_1 and X_2 are dependent

An R code for the simulation of the chain

```
X=c()
Z=c()
niter=10000
n=100
for (j in 1:niter){
x=c()
y=c()
x[1]="A"
for (i in 2:n){
  y[i]=sample(c("A","B","C"),size=1)
  change=rbinom(1,1,1/2)
  if (x[i-1]=="A") {
    if (change==0) x[i]=x[i-1]
    if (change==1) x[i]=y[i]}
  if (x[i-1]=="B") {
    if (change==0) x[i]=x[i-1]
    if (change==1) x[i]=y[i]}

  if (x[i-1]=="C") x[i]="A"}
Z[j]=x[n]==y[n]
X[j]=x[n]}
#approximated limit distribution
table(X)/niter
X
     A      B      C
0.5741 0.2870 0.1389

#approximated probability taht the two piantings are in the same room
sum(Z==1)/niter

0.6157
```