Stochastic Processes

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Surname and name ______________________________

Identification number ______________________________

Time: 2 hours to solve at least 2 exercises

1. Let X_1 and X_2 be independent discrete uniform random variables assuming values on $\{1,2,3,4,5\}$ Let $Y = \max(X_1, X_2)$ be the maximum of (X_1, X_2)

   A. Find the distribution of Y

$$P(Y \leq y) = P(X_1 \leq y)^2 = \begin{cases} (1/5)^2 & y = 1 \\ (2/5)^2 & y = 2 \\ (3/5)^2 & y = 3 \\ (4/5)^2 & y = 4 \\ 1 & y = 5 \end{cases}$$

Then

$$P(Y = j) = \left(\frac{j}{5}\right)^2 - \left(\frac{j-1}{5}\right)^2$$

   B. Find the expected value of Y

$$E(Y) = 1\frac{1}{25} + 2\frac{3}{25} + 3\frac{5}{25} + 4\frac{7}{25} + 5\frac{9}{25} = 3.8$$

   C. Find $E(Y|X_2 = 2)$

   $Y|X_2 = 2 \sim \{2,3,4,5\}$. Moreover,

$$\begin{aligned} P(Y = 2|X_2 = 2) &= P(X_1 \leq 2|X_2 = 2) = 2/5 \\ P(Y = 3|X_2 = 2) &= P(X_1 = 3|X_2 = 2) = 1/5 \\ P(Y = 4|X_2 = 2) &= P(X_1 = 4|X_2 = 2) = 1/5 \\ P(Y = 5|X_2 = 2) &= P(X_1 = 5|X_2 = 2) = 1/5 \end{aligned}$$

$$E(Y|X_2 = 2) = 2\frac{2}{5} + \frac{3+4+5}{5} = \frac{16}{5} = 3.2$$

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2. Let (X, Y) be a joint random variable such that $P(X = 1) = \omega$ and $Y|X = 1$ is a Bernoulli random variable with probability $1 - e^{-\lambda}$ while $Y|X = 0$ is Poisson (λ)

A. Find the expected value of Y

$E(Y) = \omega(1 - e^{\lambda}) + (1 - \omega)\lambda$

B. Find the distribution of Y

$$P(Y = 0) = \omega e^{-\lambda} + (1 - \omega)e^{-\lambda} = e^{-\lambda}$$
$$P(Y = 1) = \omega(1 - e^{-\lambda}) + (1 - \omega)\lambda e^{-\lambda}$$
$$P(Y = y) = (1 - \omega)\frac{e^{-\lambda}\lambda^y}{y!} \quad y > 1$$

C. Find $P(X = 1|Y = 1)$

$$P(X = 1|Y = 1) = \frac{P(Y = 1|X = 1)P(X = 1)}{P(Y = 1)} = \frac{\omega\lambda e^{-\lambda}}{= \omega(1 - e^{-\lambda}) + (1 - \omega)\lambda e^{-\lambda}}$$

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3. Let (X, Y) be a random variable with density

$$f(x, y) = \begin{cases} k(x + y) & 0 \leq x \leq 3, \quad 0 \leq y \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

A. Find the value of k.

$k = 1/27$

B. Calculate $P(X > 2Y)$

$5/24$

C. Say if X and Y are independent

No

D. Find the distribution of $X|Y = 2$

$f(x|Y = 2) = (x + 2)/10.5$

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4. Consider the random vector (X, Y) with density

$$f(x, y) = \begin{cases} 2\exp(x + y) & x \leq y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

A. Find the marginal density of X and Y

$$f(x) = \int_x^0 2e^{x+y}dy = 2e^x(1 - e^x) \quad x \in (-\infty, 0)$$

$$f(y) = \int_{-\infty}^y 2e^{x+y}dy = 2e^{2y} \quad y \in (-\infty, 0)$$

B. Find the expected value $E(X|Y=y)$

$$E(X|Y=y) = -(y+1)$$

C. Find the expected value of $Z = XY$

$$E(XY) = E(E(XY|Y)) = E(YE(X|Y)) = E(-Y(Y+1)) = -E(Y^2)-E(Y) = 1/2+1/2 = 1$$

D. Find the distribution of $W = Y - X$

$$\begin{aligned} P(W < v) &= P(Y < v + X) = 1 - P(Y > v + X) \\ &= 1 - \int_{-\infty}^{-v} 2e^x \int_{v+x}^{0} e^y dy\, dx = 1 - e^{-v} \end{aligned}$$

$W \sim (Exp(1))$

FINE COMPITO
