Stochastic Processes

Written exam July 2021

Surname and name:

–

Solve at least 2 of the following 5 exercises

1.1 A census shows that in the USA, 80 % of the daughters of working women also have a steady job, while only 30 % of the daughters of unemployed women have a steady job.

A. Write the transition matrix for the Markov process describing the working status of an american woman

B. In the lung run, what is the proportion of american women with a steady job

Let X_n be the working status of an american woman. If her mother had a steady job, we can write that

$$P(X_n = steady | X_{n-1} = steady) = 0.8$$

where X_{n-1} is the working status of her mother. Similarly

$$P(X_n = steady | X_{n-1} = unemployed) = 0.3$$

. The transition matrix is then

$$P = \begin{pmatrix} & S & U \\ S & 0.8 & 0.2 \\ U & 0.3 & 0.7 \end{pmatrix}$$

In the long run the proportions (see example 12.15 of the textbook) are

$$\pi_S = 0.3/(0.2+0.3) = 3/5 \quad \pi_U = 0.2/(0.2+0.3) = 2/5$$

1.2 Bob eats his lunch every day near his office. He goes to three places: a Chinese restaurant (C), a Mexican Tacos (T) and a salad place (I). How he chooses where to go depends on how much he ate the day before according to the transition matrix

$$\begin{pmatrix} & C & T & I \\ C & .15 & .6 & .25 \\ T & .4 & .1 & .5 \\ I & .1 & .3 & .6 \end{pmatrix}$$

Last Monday he ate Tacos.

A. What is the probability that Bob will eat in each one of the three places next Wednesday (that is after four days)

B. In the long run, what is the distribution of the three restaurants

The initial distribution is $\pi_0 = (0, 1, 0)$, then the distribution for the first day is

$$\pi_1 = \pi_0 P = (0.4, 0.1, 0.5)$$

Then we have

$$\begin{aligned} \pi_2 &= \pi_1 P = (0.15, 0.4, 0.45) \\ \pi_3 &= \pi_2 P = (0.2275, 0.265, 0.5075) \\ \pi_4 &= \pi_3 P = (0.191, 0.315, 0.494) \end{aligned}$$

The limit distribution π satisfies $\pi = \pi P$, then we have

$$\begin{cases} \pi_C \cdot 0.15 + \pi_T \cdot 0.4 + \pi_I \cdot 0.1 &= \pi_c \\ \pi_C \cdot 0.6 + \pi_T \cdot 0.1 + \pi_I \cdot 0.3 &= \pi_T \\ \pi_C + \pi_T + \pi_I &= 1 \end{cases}$$

that is by taking $\pi_I = 1 - \pi_C - \pi_T$ in the firsts two equations

$$\begin{cases} -\pi_C \cdot 0.95 + \pi_T \cdot 0.3 &= -0.1 \\ \pi_C \cdot 0.3 - \pi_T \cdot 1.2 &= -0.3 \\ \pi_C + \pi_T + \pi_I &= 1 \end{cases}$$

By the Cramer's rule we have

$$\pi_C = \frac{(-0.1) \cdot (-1.2) - (0.3) \cdot (-0.3)}{(-0.95) \cdot (-1.2) - (0.3 \cdot 0.3)} = \frac{0.21}{1.05} = 0.2$$

$$\pi_T = \frac{(-0.95) \cdot (-0.3) - (-0.1) \cdot 0.3)}{(-0.1) \cdot (-1.2) - (0.3) \cdot (-0.3)} = \frac{0.315}{1.05} = 0.3$$

and $\pi_I = 1 - 0.2 - 0.3 = 0.5$

---

2 The joint density of the double random variable (X, Y) is

$$f_{X,Y}(x, y) = Cy \exp\{-xy\} \quad 0 < x < 1;\, 0 < y < 1$$

and 0 otherwise.

A. Find the constant C such that $f_{X,Y}$ is a density

$$1 = C \int_0^1 \int_0^1 ye^{-xy} dx\, dy = C \int_0^1 1 - e^{-y} dy = C \left[y + e^{-y}\right]_0^1 = e^{-1}$$

Then $C = e$

B. Calculate $P\left(1/2 < X < 1, 0 < y < 1/2\right)$

$$\begin{aligned} P\left(1/2 < X < 1, 0 < y < 1/2\right) &= e\int_0^{1/2}\int_{1/2}^1 ye^{-xy}dx\;dy = e\int_0^{1/2} -e^{-xy}\Big|_{1/2}^1 dy \\ &= e\int_0^{1/2} -e^{-y}+e^{-y/2}\;dy = e\left[e^{-y}\Big|_0^{1/2} - 2e^{-y/2}\;\Big|_0^{1/2}\right] \\ &= \ldots = e\left(1-2e^{-1/4}+e^{-1/2}\right) \end{aligned}$$

C. Find the marginal density of X and Y

$$\begin{aligned} f(x) &= \int_0^1 e\,y\,e^{-xy}dy \overset{by\ parts}{=} e\left[-\frac{y}{x}e^{-xy}\;\Big|_{y=0}^{y=1} + \int_0^1 \frac{1}{x}e^{-xy}\,dy\right] \\ &= e\left[-\frac{1}{x}e^{-x} - \frac{1}{x^2}e^{-xy}\Big|_{y=0}^{y=1}\right] = \ldots = \frac{e}{x^2}(1-e^{-x}-xe^{-x}) \end{aligned}$$

$$f(y) = e\int_0^1 ye^{-xy}dx = -e\;e^{-xy}\Big|_0^1 = e(1-e^{-y})$$

---

3 There is a bank with only two clerks. Three customers arrive, Anna, Bruna, and Cinzia, one after the other and enter in this order. Anna and Bruna go to the two clerks while Cinzia waits for the first clerk available.

Suppose that the two clerks work independently and that for the first clerk the duration times of each operation are independent Exponential random variable with mean equal to 3 minutes while for the other clerk the duration times are independent Exponential with mean equal to 6 minutes.

A. Find the expected value of the random variable { Time spent by Cinzia inside the bank}?[1]

Let X_i for $i = A, B, C$ be the operation time of A, B, C. We know that $X_A \sim Exp(\lambda_1)$ and $X_B \sim Exp(\lambda_2)$. Given $X_A < X_B$ $X_C \sim Exp(\lambda_1)$ otherwise $X_C \sim Exp(\lambda_2)$ Note also that

$$\begin{aligned} P(X_B > X_A) &= E(E(I_{X_B<X_A}|X_A)) = E(P(X_B > X_A|X_A)) = E(e^{\lambda_2 X_A}) \\ &= \int_0^\infty e^{\lambda_2 x}\lambda_1 e^{\lambda_1 x}dx = \frac{\lambda_1}{\lambda_1+\lambda_2} \end{aligned}$$

and

$$\begin{aligned} E(X_C) &= E(X_C|X_A > X_B)P(X_A > X_B) + E(X_C|X_B > X_A)P(X_B > X_A) \\ &= \frac{1}{\lambda_2}\frac{\lambda_2}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1}\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{2}{\lambda_1+\lambda_2} = \frac{2}{1/3+1/6} = 4 \end{aligned}$$

[1]Consider also the duration of her bank operation ...

Now, let T_C the time spent by Cinzia inside the bank

$$T_C = \min\{X_A, X_B\} + X_C$$

Since

$$P(\min\{X_A, X_B\} > t) = e^{-\lambda_1 t} e^{-\lambda_2 t} = e^{-(\lambda_1+\lambda_2)t},$$

$$\min\{X_A, X_B\} \sim Exp(\lambda_1 + \lambda_2).$$

and

$$E(\min\{X_A, X_B\}) = \frac{1}{\lambda_1 + \lambda_2} = \frac{1}{1/3 + 1/6} = 2.$$

Finally

$$E(T_C) = E(\min\{X_A, X_B\}) + E(X_C) = 2 + 4 = 6$$

B. What is the average amount of time it takes for all three customers to be out of the bank?

Note that the total amout of time spent by the three customers inside the bank is

$$T = \min\{X_A, X_B\} + \max\{X_C, W\}$$

where

$$W = \max\{X_A, X_B\} - \min\{X_A, X_B\}.$$

In fact $\min\{X_A, X_B\}$ is the starting time for Cinzia to be served, W is the extra time for the first two customers to finish after Cinzia started her operation.

Conditionally on $X_A < X_B$ we have

- $X_C \sim X'_A$ where X'_A is Exp(λ_1).
- $W = (X_B - X_A) \sim X'_B$ where X'_B is Exp(λ_2) *Consequence of the memoryless property of the exponential distribution. The formal proof is*

$$\begin{aligned} P(X_B - X_A < t | X_A < X_B) &= \frac{P(X_A < X_B < X_A + t)}{P(X_A < X_B)} \\ &= \frac{\int_0^\infty \lambda_1 e^{-\lambda_1 x_A} \int_{x_A}^{x_A+t} \lambda_2 e^{-\lambda_2 x_B} dx_B \, dx_A}{\lambda_1/(\lambda_1+\lambda_2)} \\ &= \frac{\int_0^\infty \lambda_1 e^{-\lambda_1 x_A} (e^{\lambda_2 x_A} - e^{-\lambda_2(x_A+t)}) dx_A}{\lambda_1/(\lambda_1+\lambda_2)} \\ &= .... = \frac{(1 - e^{-\lambda_2 t})\lambda_1/(\lambda_1+\lambda_2)}{\lambda_1/(\lambda_1+\lambda_2)} = 1 - e^{-\lambda_2 t} \end{aligned}$$

- $T = \min\{X_A, X_B\} + \max\{X'_A, X'_B\}$

Conditionally on $X_B < X_A$ we have

- $X_C \sim X'_B$ where X'_B is Exp(λ_2).
- $W = (X_A - X_B) \sim X'_A$ where X'_A is Exp(λ_1)
- $T = \min\{X_A, X_B\} + \max\{X'_B, X'_A\}$

Then we have

$$T = \min\{X_A, X_B\} + \max\{X'_A, X'_B\}$$

also unconditionally.

It remains to calculate $E(\max\{X'_A, X'_B\})$

$$\begin{aligned} E(\max\{X'_A, X'_B\}) &= \int_0^\infty s\,\frac{d}{ds}(1-e^{-\lambda_1 s})(1-e^{-\lambda_2 s})ds \\ &= \int_0^\infty s\,(\lambda_1 e^{-\lambda_1 s} + \lambda_2 e^{-\lambda_2 s} - (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)s})ds \\ &= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2} = 3+6-2 = 7 \end{aligned}$$

Finally

$$E(T) = E(\min\{X_A, X_B\}) + E(\max\{X'_A, X'_B\}) = 2+7 = 9$$

C. What is the probability that Anna, Bruna and Cinzia respectively will be the last to leave the bank?

Let A be the event "Anna is the last to leave the bank". Note that

$$P(A|X_A < X_B) = 0$$

(if Anna finishes before Bruna she cannot be the last) and

$$P(A|X_A > X_B) = P(X'_A > X'_B)$$

(In fact in this case the duration time for Cinzia is $X'_B \sim Exp(\lambda_2)$ and the extra time for Anna to finish her operation when Cinzia starts is $X'_A \sim Exp(\lambda_1)$. Then

$$\begin{aligned} P(A) &= P(A|X_A > X_B)P(X_A > X_B) + P(A|X_A < X_B)P(X_A < X_B) \\ &= P(A|X_A > X_B)P(X_A > X_B) \\ &= P(X'_A > X'_B)P(X_A > X_B) = \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2 = \left(\frac{1/6}{1/3+1/6}\right)^2 = (1/3)^2 = 1/9 \end{aligned}$$

Similarly

$$P(B) = ... = P(X'_B > X'_A)P(X_B > X_A) = ... = 4/9$$

and $P(C) = 1 - P(A) - P(C) = 4/9$

---

4 N dice are rolled where N is a discrete Uniform random variable that takes values on $\{1, 2, \dots, 10\}$ Let S_i be the result for the ith die and let $S = \sum_{i=1}^N S_i$ be the total sum.

A. Calculate $P(N = 2|S = 4)$

B. Calculate $P(S = 4|N \text{ even })$

C. Calculate $P(S = 4, S_1 = 1|N = 2)$

D. Calculate $E(S)$

(a) $P(S = 4|N = 1) = 1/6$, $P(S = 4|N = 2) = 3/6^2$, $P(S = 4|N = 3) = 3/6^3$ $P(S = 4|N = 4) = 1/6^4$, $P(S = 4|N > 4) = 0$

$$P(N = 2|S = 4) = \frac{P(S = 4|N = 2)P(N = 2)}{P(S = 4)} = \frac{(1/10)(3/6^2)}{(1/10)(1/6 + 3/6^2 + 3/6^3 + 1/6^4)} \approx 0.315$$

(b)

$$\begin{aligned} P(S = 4|N\,pari) &= \frac{P(S = 4 \cap [N = 1 \cup N = 2])}{P(Neven)} \\ &= \frac{P(S = 4 \cap N = 2) + P(S = 4 \cap N = 4)}{1/2} \\ &= \frac{(1/10)(3/6^2) + (1/10)(1/6^4)}{1/2} \approx 0.004 \end{aligned}$$

(c)

$$P(S = 4, S_1 = 1|N = 2) = 1/36$$

(d)

$$E(S) = E(\sum_{i=1}^{N} S_i) = E(N)E(S_i) = 5.5 \cdot 3.5 = 19.25$$

---

5 The probability that any child in a certain family will have blue eyes is 1/4, and this feature is inherited independently by different children in the family. Suppose that the family has five children

A. What is the most probable number of children with blue eyes?

Let X be the number of children with blue eyes. X is Binomial (5,1/4).

$$\begin{aligned} P(X = 0) &= \binom{5}{0} 0.25^0 0.75^5 = 0.237 \\ P(X = 1) &= \binom{5}{1} 0.25^1 0.75^4 = 0.396 \\ P(X = 2) &= \binom{5}{2} 0.25^2 0.75^3 = 0.264 \\ P(X = 3) &= \binom{5}{3} 0.25^3 0.75^3 = 0.088 \\ P(X = 4) &= \binom{5}{4} 0.25^4 0.75^1 = 0.015 \\ P(X = 5) &= \binom{5}{5} 0.25^5 0.75^0 = 0.001 \end{aligned}$$

hence with 5 childrens the mode is 1

B. If it is known that at least one of these children has blue eyes, what is the probability that at least three of the children have blue eyes?

$$P(X \geq 3|X \geq 1) \quad = \quad \frac{P(X \geq 3 \cap X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 3)}{P(X \geq 1)} = \frac{0.088 + 0.015 + 0.001}{1 - 0.237} = \frac{0.104}{0.763} = 0.136$$

C. If it is known that the youngest child in the family has blue eyes, what is the probability that at least three of the children have blue eyes?

Note that $X = Y_1 + Y_2 + Y_3 + Y_4 + Y_5$ where the r.v. Y_i indicates if the ith child has blue eyes .

By the independence of the random variables Y_i the requested probability is

$$\begin{aligned} P(X \geq 3|Y_5 = 1) &= P(Y_1 + Y_2 + Y_3 + Y_4 + Y_5 \geq 3|Y_5 = 1) \\ &= P(Y_1 + Y_2 + Y_3 + Y_4 \geq 3 - 1|Y_5 = 1) \\ &= P(Y_1 + Y_2 + Y_3 + Y_4 \geq 2) \\ &= \sum_{k=2}^{4} \binom{4}{k} 0.25^k 0.75^{4-k} = 0.211 + 0.047 + 0.004 = 0.262 \end{aligned}$$

D. Explain why the answer in part (C) is different from the answer in part (B).

Note that the conditioning events $X \geq 1$ and $Y_5 = 1$ are different and for solving part C we exploited the indipendence between Y_5 and $Y_1 + Y_2 + Y_3 + Y_3 + Y_4$

FINE COMPITO
