# STOCHASTIC PROCESSES

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**Time: 2 hours. Solve at least 2 exercises.**

1. Let $(X, Y)$ be a random variable with joint density

$$f_{X,Y}(x,y) = \begin{cases} 4x^2y + 2y^5 & 0 \le x \le 1; 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Consder $Z = X + Y^2$ and $W = X - Y^2$

A. Find the joint density of $(Z, W)$ and specify the set where the density is positive

Note that $Z+W = 2X$ and $Z-W = 2Y^2$, then $X = \frac{1}{2}(Z+W)$ and $Y = \frac{1}{\sqrt{2}}\sqrt{Z-W}$. The density is positive on the set

$$Q = \{(z,w) : 0 < z < 1, -z < w < z\} \cup \{(z,w) : 1 < z < 2, -2+z < w < 2-z\}$$

The Jacobian of the transformation is

$$J(z,w) = \left| \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}(z-w)^{1/2}} & -\frac{1}{2\sqrt{2}(z-w)^{1/2}} \end{array} \right| = \frac{1}{2^{3/2}} \frac{1}{(z-w)^{1/2}}.$$

The density is

$$\begin{aligned} f_{Z,W}(z,w) &= f_{X,Y}(x = (z+w)/2, y = \sqrt{(z-w)/2}) J(z,w) \\ &= \frac{1}{8}(3z^2 + 3w^2 - 2zw) \quad (z,w) \in Q \end{aligned}$$

B. Find the marginal density of $Z$.

$$f_Z(z) = \left\{ \begin{array}{ll} z^3 & 0 < z < 1 \\ \frac{1}{8}(6z^2(2-z) + 2(2-z)^3) & 1 < z < 2 \end{array} \right.$$

In fact for $0 < z < 1$

$$f_Z(z) = \int_{-z}^{z} \frac{1}{8}(3z^2 + 3w^2 - 2zw)dw = \frac{1}{8}(6z^3 + 2z^3 + 0) = z^3$$

For $1 < z < 2$

$$f_Z(z) = \int_{-2+z}^{2-z} \frac{1}{8}(3z^2 + 3w^2 - 2zw)dw = ... = \frac{1}{8}(6z^2(2-z) + 2(2-z)^3)$$

2. You throw three regular dice. Let $S$ be the random variable indicating the total score

   A. Indicate the set of values taken by $S$. Calculate the expected value and the variance of $S$.

   Indicate with $X_A, X_B$ and $X_C$ the results of the three dice. $S = X_A + X_B + X_C$
   The image of $S$ is the set $\mathcal{S} = \{3, 4, \ldots, 18\}$

$$E(S) = E(X_A) + E(X_B) + E(X_C) = 3\sum_{i=1}^{6} i\frac{1}{6} = 3\frac{21}{6} = 10.5$$

$$E(X_A^2) = \sum_{i=1}^{6} i^2\frac{1}{6} = \frac{91}{6}$$

$$Var(S) = Var(X_A) + Var(X_B) + Var(X_B) = 3Var(X_a) = 3(91/6 - 21^2/6^2) = 105/12$$

   B. Find $P(S = 10)$

$$\begin{aligned} P(S = 10) &= \sum_{j=1}^{6} P(S = 10|X_A = j)P(X_A = j) \\ &= \frac{1}{6}\sum_{j=1}^{6} P(X_A + X_B + X_C = 10|X_A = j) \\ &= \frac{1}{6}\sum_{j=1}^{6} P(X_B + X_C = 10 - j|X_A = j) \\ &= \frac{1}{6}\sum_{j=1}^{6} P(X_B + X_C = 10 - j) \\ &= \frac{1}{6}\frac{1}{36}(4 + 5 + 6 + 5 + 4 + 3) = \frac{27}{216} \end{aligned}$$

   C. Find $P(S = 10|\text{Dice A is equal to } 4)$

$$\begin{aligned} P(X_A + X_B + X_C = 10|X_A = 4) &= P(X_B + X_C = 6|X_A = 4) \\ &= P(X_B + X_C = 6) = \frac{5}{36} \end{aligned}$$

3. The duration time in days of an electronic component is a Gamma random variable $X$ with parameter $\alpha = 5$ and $\theta = 1/10$, i.e.

$$f_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x}, \qquad x > 0.$$

When the component breaks, it is immediately replaced by a new one.

A. Use the central limit theorem to obtain the probability that 40 components are enough to last at least 6 years (assume that each year has 365.25 days)

The mean of $X$ is $\mu = E(X) = \alpha/\theta = 5 \cdot 10 = 50$, the standard deviation of $X$ is $\sigma = \sqrt{\alpha/\theta^2} = \sqrt{5100} = 22.36$. Let $S = \sum_{i=1}^{40} X_i$ where $X_i$ are i.i.d. Gamma$(\alpha, \theta)$

$$\begin{aligned} P(S > 6 \text{ years}) &= P(S > 6 \cdot 365.25) = P(S > 2191.5) \\ &= P\left(\sum_{i=1}^{40} X_i > 2191.5\right) = P\left(\frac{\sum_{i=1}^{40} X_i - 40 * 50}{\sqrt{40} \cdot 22.36} > \frac{2191.5 - 40 * 50}{\sqrt{40} \cdot 22.36}\right) \\ &= 1 - \Phi(1.35) = 0.0885 \end{aligned}$$

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s=c()
nsim=10^5
for (i in 1:nsim) s[i]=sum(rgamma(40,shape=5,rate=1/10))
sum(s>6*365.25)/nsim
0.0881
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B. By the central limit theorem, find an expression for the expected value of the number of components which will be used to last at least 6 years. (Use the fact that for a positive r.v $Y$, $E(Y) = \sum_{y=0}^{\infty} P(Y > y)$)

Let $N$ be the number of components which will be used to last at least 6 years.

$$N > j \Leftrightarrow \sum_{i=1}^{j} X_i < 2191.5$$

Note that $P(N > 0) = 1$ and for $j \geq 1$

$$\begin{aligned} P(N > j) &= P\left(\sum_{i=1}^{j} X_i < 2191.5\right) \\ &= P\left(\frac{\sum_{i=1}^{j} X_i - j * 50}{\sqrt{j} \cdot 22.36} < \frac{2191.5 - j * 50}{\sqrt{j} \cdot 22.36}\right) \\ &= \Phi\left(\frac{2191.5 - j * 50}{\sqrt{j} \cdot 22.36}\right) \end{aligned}$$

$$E(N) = 1 + \sum_{j=1}^{\infty} \Phi\left(\frac{2191.5 - j * 50}{\sqrt{j} \cdot 22.36}\right)$$

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A function to simulate the r.v. N
N=function(n) {N=c()
+   for (i in 1:n) N[i]=which(cumsum(rgamma(1000,shape=5,rate=1/10))>6*365.25)[1]
+   return(N)}
>   mean(N(10000))
[1] 44.4246
A function to calculate  P(N>j) by the  CLT
> pngj=function(j) pnorm( (2191.5-j*50)/(sqrt(j)*22.36))
> 1+ sum(pngj(1:10000))
[1] 44.42999
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4. Let $(N, Y)$ be such that $Y|N = n \sim Binomial(n, p)$ and $N \sim \text{Poisson}(\lambda)$.

   A Write the probability mass function of the double random variable. Specify the points where the probability mass function is positive

   B Find the marginal distribution of $Y$

   C Find the conditional distribution of $N|Y = y$

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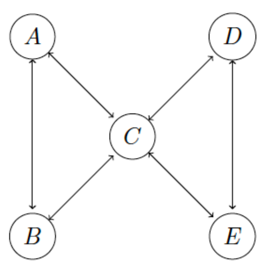
5. A particle follows a random walk on the set described in the following figure




At time $n$, the particle moves to one of the adjacent nodes with equal probability

A. Write the transition matrix of the chain

$$P = \begin{pmatrix} & A & B & C & D & E \\ A & 0 & 1/2 & 1/2 & 0 & 0 \\ B & 1/2 & 0 & 1/2 & 0 & 0 \\ C & 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ D & 0 & 0 & 1/2 & 0 & 1/2 \\ E & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

B. Find the invariant distribution $\pi = (1/6, 1/6, 1/3, 1/6, 1/6)$

C. Suppose that $X_0 = C$. Find the expected value of the first passage time in $A$

Note that by symmetry $E(T_A|X_0 = D) = E(T_A|X_0 = E)$

Moreover

$$\begin{aligned} E(T_A|X_0 = D) &= E(T_A|X_1 = C)P(X_1 = C|X_0 = D) + E(T_A|X_1 = E)P(X_1 = E|X_0 = D) \\ &= \frac{1}{2}(1 + E(T_A|X_0 = C)) + \frac{1}{2}(1 + E(T_A|X_0 = E)) \\ &= \frac{1}{2}(1 + E(T_A|X_0 = C)) + \frac{1}{2}(1 + E(T_A|X_0 = D)) \\ &= 1 + \frac{1}{2}E(T_A|X_0 = C) + \frac{1}{2}E(T_A|X_0 = D)) \end{aligned}$$

Then

$$E(T_A|X_0 = D) = 2 + E(T_A|X_0 = C)$$

In addition

$$E(T_A|X_0 = B) = \frac{1}{2} \cdot 1 + \frac{1}{2}(1 + E(T_A|X_0 = C)) = 1 + \frac{1}{2}E(T_A|X_0 = C)$$

Hence

$$\begin{aligned} E(T_A|X_0 = C) &= \frac{1}{4} \cdot 1 + \frac{1}{4}(1 + E(T_A|X_0 = B)) + \frac{1}{4}(1 + E(T_A|X_0 = D)) + \frac{1}{4}(1 + E(T_A|X_0 = E)) \\ &= \frac{1}{4} \cdot 1 + \frac{1}{4}(1 + E(T_A|X_0 = B)) + \frac{1}{2}(1 + E(T_A|X_0 = D)) \\ &= \frac{1}{4} \cdot 1 + \frac{1}{4}(1 + 1 + \frac{1}{2}E(T_A|X_0 = C)) + \frac{1}{2}(1 + 2 + E(T_A|X_0 = C)) \\ &= \frac{5}{8}E(T_A|X_0 = C) + \frac{9}{4} \end{aligned}$$

That is

$$E(T_A|X_0 = C) = 6$$

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FINE COMPITO

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