STOCHASTIC PROCESSES

29.10.2021

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Surname and name:

Identification number:

Time: 2 hours. Solve at least 2 exercises.

1. Let X and Y be two random variables such that the marginal density of X is

$$f_X(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and the conditional density $Y|X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} 3y^2/x^3 & 0 \leq y < x \\ 0 & \text{otherwise} \end{cases};$$

Find

A. The image of the r.v. Y.

B. The marginal density of Y.

C. The conditional density of $X|Y$.

D. The covariance between X and Y.

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2. Three children, A, B, and C throw a ball between them. When A has the ball, he throws it to B with probability 0.2 or to C with complementary probability. When B has the ball, he throws it to A with probability 0.6 or to C with complementary probability. Finally, when C has the ball, he throws it to A or B with the same probability. Let X_n be the children throwing the ball at time $n = 0, 1, \ldots$

A. Consider the process as a Markov chain and write the transition matrix of the process.

B. Suppose that the children throwing the ball at time 0 is uniformly selected, who has the higher probability of throwing the ball when $n = 2$

C. Answer to the same question when $n = 100$

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3. You have three red balls (R) and three white balls (B). The balls are randomly thrown into three boxes (for each throw the three boxes have the same probability to receive the ball and the throws are independent)

   A. What is the probability that each box contains exactly a red ball and a white ball?

   B. What is the probability that all the three red balls are thrown the same box?

   C. Now modify the experiment and suppose to have three balls of the same color and five boxes. What is the probability that no box contains more than one ball?

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4. Let $X_1, X_2, \ldots, X_n$ be n independent random variables such that for $j = 1, \ldots, n$,

$$EX_j = \mu; \qquad \mathrm{Var}(X) = \sigma^2.$$

   Let $\bar{X} = \sum_{j=1}^{n} X_j/n$ be the sample mean

   Find

   A. Mean and variance of $\bar{X}$

   B. The correlation coefficient between $\bar{X}$ and X_1.

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5. Let X be a random variable with distribution function

$$F_X(t) = \begin{cases} 0 & t \leq 0 \\ \frac{2}{5}t & 0 < t \leq 1 \\ \frac{3}{5}t - \frac{1}{5} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}$$

   A. Verify that $F_X(\cdot)$ is a distribution function

   B. Verify that X is a continuous random variable and find the density

   C. Without doing any calculations find $P(X \leq 1.5)$.

   D. Consider $Y = X^2/4$. Find the distribution of Y

FINE COMPITO
