STOCHASTIC PROCESSES

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Surname and name:

Identification number:

Time: 2 hours. Solve 2 exercises.

1. Three students, A, B, C, have equal claims to receive an award. Then, they decide that each will toss a coin, and that the man whose coin falls unlike the other two wins. If all three coins fall alike, they toss again.

   A. Describe a sample space for the result of the first toss of the three coins, and assign probabilities to its elements.

   B. What is the probability that A wins on the first toss? That B does? That C does? That there is no winner on the first toss?

   C. Given that there is a winner on the first toss, what is the probability that it is A?

   D. Find the marginal probability that A eventually wins the award.

   E. Given that A wins, what is the distribution of the number of tosses which will be necessary to end the game?

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2. You choose at random a point (U, V) inside a rectangle whose sides have the lengths 2 and 3. That is (U, V) is uniformly distributed inside this rectangle. Let X be the distance of the point form the closest side of the rectangle.

   A. Find the support of X and the cumulative distribution function of X

   B. Compute EX

   C. Find the mean of the random variable Y given by the area of the square with side X

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3. The discrete joint distribution of the lifetimes X and Y of two connected components in a machine can be modelled by

$$P(X = k, Y = j) = \frac{1}{e^2} \frac{1}{k!(j-k)!}$$

   for $k = 0, 1, 2, \ldots$ and $j = k, k+1, k+2, \ldots$

   A. Find the marginal distribution of X and Y

   B. Find the joint distribution of X and $Z = Y - X$. Are X and Z independent?

   C. Find the correlation between X and Y

4. Initially a box contains 3 red balls and 2 black balls. The box is sequentially modified accordingly to the following rules. We extract a ball, if it is black we remove it. If the ball is red we put it again in the box and add another black ball. Moreover, when there are not black balls anymore, we do not make any extraction but we add two black balls. Similarly when there are four black balls, we do not make any extraction but we remove two black balls from the box. Let X_n be the number of black balls after n manipulations.

   A. Find the transition matrix of the chain

   B. Find the distribution of X_1 and X_2

   C. Is the chain irreducible? In case find the limit distribution of the chain.

   D. Now suppose that when the chain reaches the states 0 or 4 it remains stuck in these states. Is this chain irreducible? Finally suppose that when the number of black balls is 0 then only one black ball is added and when there are 4 black balls only one black ball is removed. In this case, do we have a limit distribution?

END OF THE EXAM
