STOCHASTIC PROCESSES

Prof. Andrea Tancredi

Surname and name:

Identification number:

Time: 2 hours. Solve 2 exercises.

1. Three players, A, B, and C, take turns to roll a die; they do this in the order ABCABCA...

   A. Describe the sample space of the first turn reporting the probability for each possible outcome

   B. Calculate the probability that, of the three players, A is the first to throw a 6

   C. Calculate the probability that, of the three players, A is the first to throw a 6, B the second, and C the third

   D. Calculate the probability that the first 6 to appear is thrown by A, the second 6 to appear is thrown by B, and the third 6 to appear is thrown by C

---

2. Let X, Y be a random variable uniformly distributed on the triangle with vertices $(0,0), (1,1), (0,1)$

   A. Write the joint density of X, Y. Find the marginal densities of X and Y and calculate $cov(X,Y)$

   B. Find the density of $W = \max\{X, Y\}$

   C. Find the joint density of U, Z where $U = X + Y$ and $Z = X - Y$

   D. Calculate the covariance between U and Z. Are U and Z independent?

---

3. Let X_1 and X_2 be independent, idenntically distributed random variables with density

$$f(x) = \begin{cases} 2x/a^2 & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

   where a is a positive constant. Define new random variables Y and Z as

$$Y = \max(X_1, X_2); \qquad Z = \min(X_1, X_2).$$

   A. Compute mean and variance of X_1

   B. Compute the distribution and the mean of Y and Z.

C. Prove that the joint density of (Y, Z) is

$$g_{YZ}(y,z) = \begin{cases} 8yx/a^4 & 0 < z < y < a \\ 0 & \text{othrwise} \end{cases}$$

Hint: Start by computing $P(Y \leq y \cap Z > z)$

D. Find the mean of $U = Y - Z$.

4. The buses of an urban line pass to a specific stop every 15 minutes starting at 7 (therefore at 7, 7:15, 7:30 and so on). Sara arrives at the bus stop in an instant that distributes itself as a uniform random variable in the interval $[7:00, 7:30]$.

A) Calculate the probability that Sara will wait less then 5 minutes

B) Calculate the probability that Sara will wait more that 10 minutes

C) The mean waiting time at the bus stop

D) Now suppose that bus departures are not deterministic but random, with waiting times in minutes between one bus and another distributed as independent and identically distributed exponential random variables with $\lambda = 1/15$. Calculate the probability that Sara will wait less then 5 minutes

5. A cat and a mouse move independently back and forth between two rooms. At each time step, the cat moves from the current room to the other room with probability 0.8. Starting from room 1, the mouse moves to room 2 with probability 0.3 (and remains otherwise). Starting from room 2, the mouse moves to room 1 with probability 0.6 (and remains otherwise).

A. Find the stationary distributions of the cat chain and of the mouse chain.

B. Note that there are 4 possible (cat, mouse) states: both in room 1, cat in room 1 and mouse in room 2, cat in room 2 and mouse in room 1, and both in room 2. Number these cases 1, 2, 3, 4, respectively, and let Z_n be the number representing the (cat, mouse) state at time n. Explain why Z_n is still a Markov chain and find the transition matrix.

C. Suppose that the cat and the mouse at time 0 are together in one of the two rooms with the same probability. What is the probability that they are in the same room at time 2?

D. (Optional) Now suppose that the cat will eat the mouse if they are in the same room. We wish to know the expected time (number of steps taken) until the cat eats the mouse for two initial configurations: when the cat starts in room 1 and the mouse starts in room 2, and vice versa. Set up a system of two linear equations in two unknowns whose solution is the desired values.

END OF THE EXAM