STOCHASTIC PROCESSES

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Surname and name:

Identification number:

Time: 2 hours. Solve 2 exercises.

1. Three players, A, B, and C, take turns to roll a die; they do this in the order ABCABCA...

   A. Describe the sample space of the first turn reporting the probability for each possible outcome

   B. Calculate the probability that, of the three players, A is the first to throw a 6

   To be the first, the player A can throw a 6 in the first roll, or in the second roll when nobody throw a 6 in first roll, or in the third roll when nobody throw a 6 in the first two rolls and so on...

$$P(\text{A is the first}) = \frac{1}{6} + \left(\frac{5}{6}\right)^3 \frac{1}{6} + \left[\left(\frac{5}{6}\right)^3\right]^2 \frac{1}{6} + \cdots$$

$$P(\text{A is the first}) = \sum_{k=1}^{\infty} \left(\frac{5^3}{6^3}\right)^{k-1} \frac{1}{6} = \frac{6^2}{6^3 - 5^3}$$

   C. Calculate the probability that, of the three players, A is the first to throw a 6, B the second, and C the third

   Suppose that the event "*A is the first*" occurred. Conditionally on this event, B will be the second to obtain 6 if she will throw a 6 before C (note that A can still continue to obtain 6). Then, the player B can throw a 6 in the first roll after the 6 of A, or in the second roll after the 6 of A when neither B nor C throw a 6 in their first rolls, or in the third roll after the 6 of A when neither B nor C throw a 6 in their first two rolls and so on...

$$P(\text{B is the second }|\text{A is the first}) = \sum_{k=1}^{\infty} \left(\frac{5^2}{6^2}\right)^{k-1} \frac{1}{6} = \frac{6}{6^2 - 5^2}$$

   Hence

$$\begin{aligned} P(\text{A is the first, B the second, C the third}) &= P(\text{A is the first and B is the second}) \\ &= P(\text{ B is the second}|\text{A is the first})P(\text{A is the first}) \\ &= \frac{6^2}{6^3 - 5^3} \times \frac{6}{6^2 - 5^2} = \frac{216}{1001} \end{aligned}$$

D. Calculate the probability that the first 6 to appear is thrown by A, the second 6 to appear is thrown by B, and the third 6 to appear is thrown by C

When A obtains the first 6, we can think that the sequence restarts with B that now has to be the first to obtain 6. Again when B obtains the 6, the sequence starts with C who must be the first to obtain 6. Given independence we have that the answer is

$$\left(\frac{6^2}{6^3-5^3}\right)^3 = \frac{46656}{753571}$$

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2. Let X, Y be a random variable uniformly distributed on the triangle with vertices $(0,0), (1,1), (0,1)$

A. Write the joint density of X, Y. Find the marginal densities of X and Y and calculate $cov(X,Y)$ $X \sim Beta(1,2)$ $Y \sim Beta(2,1)$ cov(X,Y)=1/36

In fact

$$f_{XY}(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & otherwise \end{cases}$$

$$f_X(x) = \int_x^1 2dy = 2(1-x) \quad 0 < x < 1 \quad E(X) = \frac{1}{3}$$

$$f_Y(y) = \int_0^y 2dx = 2y \quad 0 < y < 1 \quad E(Y) = \frac{2}{3} \quad E(Y^2) = \frac{1}{2}$$

Note also that

$$X|Y = y \sim Uniform(0,y) \quad E(X|Y=y) = \frac{y}{2}$$

$$\begin{aligned} cov(X,Y) &= E(XY) - E(X)E(Y) = E(E(XY|Y)) - \frac{2}{9} \\ &= E(YE(X|Y)) - \frac{2}{9} = \frac{1}{2}E(Y^2) - \frac{2}{9} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36} \end{aligned}$$

B. Find the density of $W = \max\{X, Y\}$ $W \sim Beta(2,1)$ In fact since $Y > X$ $\max X, Y = Y$. Alternatively not that

$$\begin{aligned} P(W \le w) &= P(X \le w, Y \le w) \quad note\ that\ X, Y\ \ are\ not\ independent \\ &= \int_0^w \int_x^w 2\,dx\,dy = 2\ times\ \ area\ of\ the\ triangle\ (0,0),(w,w),(0,w) \\ &= 2\frac{w^2}{2} = w^2 \quad 0 < w < 1 \end{aligned}$$

$$f_W(w) = 2w \quad 0 < w < 1$$

Hence W has the same distribution of X, note that X and W in this case are exactly the same random variable

C. Find the joint density of U, Z where $U = X + Y$ and $Z = X - Y$ U, Z is Uniform on the triangle $(0,0),(2,0)(1,-1)$

In fact $X = (U+V)/2$ and $Y = (U-V)/2$. The random variable (U, Z) has positive density on the set

$$T^* = \{u \in (0,2), v \in (-1,0), 0 < \frac{1}{2}(u+v) < 1, 0 < \frac{1}{2}(u-v) < 1\}$$

$$T^* = \{u \in (0,2), v \in (-1,0), -u < v < 2-u, u-2 < v < u\}$$

That is on the triangle $(0,0),(2,0)(1,-1)$

Note that $|J| = 1/2$. Hence

$$f_{UZ}(u,z) = f_{XY}(x = (u+v)/2, y = (u-v)/2)|J| = 2\frac{1}{2} = 1 \quad u, v \in T^*$$

D. Calculate the covariance between U and Z. Are U and Z independent?

the covariance is 0 but U and Z are dependent

$$\begin{aligned} Cov(U,V) &= E((X+Y)(X-Y)) - E(X+Y)E(X-Y) \\ &= E(X^2 - Y^2) - (\frac{1}{3} + \frac{2}{3})(\frac{1}{3} - \frac{2}{3}) = E(X^2 - Y^2) + \frac{1}{3} \\ &= \frac{1}{6} - \frac{1}{2} + \frac{1}{3} = 0 \end{aligned}$$

U and Z are dependent. In fact the conditional density $U|V = v$ depends on v

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3. Let X_1 and X_2 be independent, identically distributed random variables with density

$$f(x) = \begin{cases} 2x/a^2 & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

where a is a positive constant. Define new random variables Y and Z as

$$Y = \max(X_1, X_2); \qquad Z = \min(X_1, X_2).$$

A. Compute mean and variance of X_1

$$E(X_1) = \int_0^a \frac{2x^2}{a^2} dx = \frac{2}{3}a \quad E(X^2) = \int_0^a \frac{2x^3}{a^2} dx = \frac{2}{4}a^2 \quad var(X_1) = E(X^2) - E(X)^2 = \frac{1}{2}a^2 - \frac{4}{9}a^2 = \frac{a^2}{18}$$

B. Compute the distribution and the mean of Y and Z.

The distribution function of X_1 is equal to

$$F_{X_1}(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{a^2} & 0 \le x < a \\ 1 & x \ge a \end{cases}$$

$$F_Y(y) = P(Y \leq y) = P(X_1 \leq y, X_2 \leq y) = P(X_1 \leq y)P(X_2 \leq y) = \begin{cases} 0 & y < 0 \\ \frac{y^4}{a^4} & 0 \leq y < a \\ 1 & y \geq a \end{cases}$$

The density of Y is then

$$f_Y(y) = \begin{cases} 4\frac{y^3}{a^4} & 0 < y < a \\ 0 & otherwise \end{cases}$$

$$F_Z(z) = 1 - P(Z > z) = 1 - P(X_1 > z, X_2 > z) = 1 - P(X_1 > z)P(X_2 > z) = \begin{cases} 0 & z < 0 \\ 1 - (1 - \frac{z^2}{a^2})^2 & 0 \leq z < a \\ 1 & z \geq a \end{cases}$$

The density of Z is then

$$f_Z(z) = \begin{cases} \frac{4z}{a^2}(1 - \frac{z^2}{a^2}) & 0 < z < a \\ 0 & otherwise \end{cases}$$

$$E(Y) = \frac{4}{5}a \quad E(Z) = \frac{8}{15}a$$

C. Prove that the joint density of (Y, Z) is

$$g_{YZ}(y, z) = \begin{cases} 8yz/a^4 & 0 < z < y < a \\ 0 & \text{othrwise} \end{cases}$$

Hint: Start by computing $P(Y \leq y \cap Z > z)$

$$\begin{aligned} P(Y \leq y, Z > z) &= P(z < X_1 \leq y, z < X_2 \leq y) = P(z \leq X_1 < y)^2 \\ &= (F_{X_1}(y) - F_{X_1}(z))^2 = \left(\frac{y^2}{a^2} - \frac{z^2}{a^2}\right)^2 \\ &= \frac{1}{a^4}(y^2 - z^2)^2 \end{aligned}$$

Then for $0 < z < y < a$

$$\begin{aligned} F_{YZ}(y, z) &= P(Y \leq y, Z \leq z) = P(Y \leq y) - P(Y \leq y, Z > z) \\ &= \frac{y^4}{a^4} - \frac{1}{a^4}(y^2 - z^2)^2 \end{aligned}$$

and

$$g_{YZ}(y, z) = \frac{\partial}{\partial y}\frac{\partial}{\partial z}F_{YZ}(y, z) = \begin{cases} 8yz/a^4 & 0 < z < y < a \\ 0 & \text{othrwise} \end{cases}$$

D. Find the mean of $U = Y - Z$.

$E(U) = E(Y - Z) = E(Y) - E(Z) = \frac{4}{5}a - \frac{8}{15}a = \frac{4}{15}a$

4. The buses of an urban line pass to a specific stop every 15 minutes starting at 7 (therefore at 7, 7:15, 7:30 and so on). Sara arrives at the bus stop in an instant that distributes itself as a uniform random variable in the interval $[7:00, 7:30]$.

A) Calculate the probability that Sara will wait less then 5 minutes Let T be the waiting time of Sara and let X be the arrival time at the bus stop. Assume that $X \sim Uniform(0, 30)$

$$P(T < 5) = P(X \in (10, 15) \cup X \in (25, 30)) = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$$

B) Calculate the probability that Sara will wait more that 10 minutes

$$P(T > 10) = P(X \in (0, 5) \cup X \in (15, 20)) = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$$

C) The mean waiting time at the bus stop

$$\begin{aligned} E(T) &= E(T|X < 15)P(X < 15) + E(T|X > 15)P(X > 15) \\ &= E(15 - X|X < 15)\frac{1}{2} + E(30 - X|X > 15)\frac{1}{2} \\ &= (15 - 7.5)\frac{1}{2} + (30 - 22.5)\frac{1}{2} = 7.5 \end{aligned}$$

D) Now suppose that bus departures are not deterministic but random, with waiting times in minutes between one bus and another distributed as independent and identically distributed exponential random variables with $\lambda = 1/15$. Calculate the probability that Sara will wait less then 5 minutes

Let $X = x$ be the Sara's arrival time at the bus stop and let W be the elapsed time between the last bus before x and the following. Moreover let $Y = y$ the time of the last bus arrival before x. Note that $y < x$ and W is exponential λ. Then

$$P(T < 5|X = x, Y = y) = P(W < x - y + 5|W > x - y)$$

and by the lack of memory of the exponential distribution the last probability is equal to

$$P(W < x - y + 5|W > x - y) = 1 - e^{-\frac{1}{15}5}$$

and does not depend on x and y. Hence

$$P(T < 5) = P(T < 5|X = x, Y = y) = 1 - e^{-\frac{1}{15}5}$$

5. A cat and a mouse move independently back and forth between two rooms. At each time step, the cat moves from the current room to the other room with probability 0.8. Starting from room 1, the mouse moves to room 2 with probability 0.3 (and remains otherwise). Starting from room 2, the mouse moves to room 1 with probability 0.6 (and remains otherwise).

   A. Find the stationary distributions of the cat chain and of the mouse chain.

      The transition matrices for the cat chain and the mouse chain are

$$\begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix} \quad \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix}$$

      The stationary distribution for the cat chain solves the system

$$\begin{cases} \pi_1 0.2 + \pi_2 0.8 & = \pi_1 \\ \pi_1 + \pi_2 & = 1 \end{cases}$$

      and it is $\pi = (1/2, 1/2)$

      The stationary distribution for the mouse chain solves the system

$$\begin{cases} \pi_1 0.7 + \pi_2 0.6 & = \pi_1 \\ \pi_1 + \pi_2 & = 1 \end{cases}$$

      and it is $\pi = (2/3, 1/3)$

   B. Note that there are 4 possible (cat, mouse) states: both in room 1, cat in room 1 and mouse in room 2, cat in room 2 and mouse in room 1, and both in room 2. Number these cases 1, 2, 3, 4, respectively, and let Z_n be the number representing the (cat, mouse) state at time n. Explain why Z_n is still a Markov chain and find the transition matrix.

      The transition matrix is

$$P = \begin{pmatrix} 0.14 & 0.06 & 0.56 & 0.24 \\ 0.12 & 0.08 & 0.48 & 0.32 \\ 0.56 & 0.24 & 0.14 & 0.06 \\ 0.48 & 0.32 & 0.12 & 0.08 \end{pmatrix}$$

   C. Suppose that the cat and the mouse at time 0 are together in one of the two rooms with the same probability. What is the probability that they are in the same room at time 2?

$$\pi_0 = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right)$$

$$\pi_1 = \pi_0 P = \left(\frac{14+48}{200}, \frac{6+32}{200}, \frac{56+12}{200}, \frac{24+8}{200}\right) = \left(\frac{62}{200}, \frac{38}{200}, \frac{68}{200}, \frac{32}{200}\right)$$

$$\pi_2 = \pi_1 P = \left(\frac{6668}{20000}, \cdots, \cdots, \frac{3368}{20000}\right)$$

      so that the probability that they are both in the same room at time 2 is 10036/20000

D. (Optional) Now suppose that the cat will eat the mouse if they are in the same room. We wish to know the expected time (number of steps taken) until the cat eats the mouse for two initial configurations: when the cat starts in room 1 and the mouse starts in room 2, and vice versa. Set up a system of two linear equations in two unknowns whose solution is the desired values.

Let t_1 be the expected time for the first configuration and t_2 be the expected time for the second configuration

The system is the following

$$t_1 = 0.12 \times 1 + 0.32 \times 1 + (1 + t_1) \times 0.08 + (1 + t_2) \times 0.48$$
$$t_2 = 0.56 \times 1 + 0.06 \times 1 + (1 + t_1) \times 0.24 + (1 + t_2) \times 0.14$$

The solution is $t_1 = 335/169$ $t_2 = 290/169$

END OF THE EXAM