Stochastic Processes

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Name and surname:

Solve 2 exercises: time 2 hours.

1. Let Z be a random variable with Uniform distribution on $(0, 2\pi)$, i.e. $Z \sim U(0, 2\pi)$. Define

$$R_1 = \cos(Z) \text{ and } R_2 = \sin(Z).$$

   A. Prove that $E(R_1 R_2) = E(R_1)E(R_2)$

   B. Prove that $\text{Var}(R_1 + R_2) = \text{Var}(R1) + \text{Var}(R_2)$

   C. Show that R_1 and R_2 are dependent, notwithstanding the previous results

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2. A box contains 8 balls enumerated from 1 to 8. You extract 4 balls without replacement

   A. Calculate the expected value of the total score obtained by summing the values of the 4 extracted balls

   B. Calculate the probability that the maximum score is 6

   C. Calculate the probability that 6 is the second higher value?

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3. Let there be given 6 empty urns, and consider a sequence of independent trials, each consisting of placing a marble in an urn chosen at random. Let X_n be the number of empty urns after n trials, $n \in \mathbb{N}$.

   A. Show that $\{X_n, n > 0\}$ is a homogeneous Markov chain.

   B. Find the transition matrix.

   C. Classify the states.

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4. Let X and Y be independent normal random variables

   A. Find the density of $Z = X^2 + Y^2$

   B. Find the joint density of $V = X + Y$ and $W = X - Y$

   C. Find the covariance between V and W

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5. A man has got 2 dogs Bruto and Pluto. Every day he takes a long walk and brings the dogs with him with the following rules

   - If Pluto takes the long walk at time t, the following day Pluto comes again with probability 0.5
   - If Bruto takes the long walk at time t, the following day Bruto comes again with probability 0.5
   - Both the dogs never spend two days without doing this long walk

   Let X_n be state indicating the dogs going out at time t where X_n can be "no dogs", "only Pluto", "only Bruto" and "Bruto and Pluto"

   A. Write the transition matrix of the chain X_n

   B. Suppose that at time 0 the man takes the long walk without the dogs, what is the probability that at time 2 he goes out with Bruto and Pluto?

   C. Find, if exists the limit distribution of the chain

   D. Find the frequency of the number of times that only one dog goes out