Stochastic Processes

Prof. Andrea Tancredi
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Name and surname:

Solve 2 exercises: time 2 hours.

1. Let (X, Y) be a bivariate continuous random variable with density

$$X|Y = y \sim \text{Exp }(y); \qquad Y \sim \text{Exp}(\nu) \quad \nu > 0.$$

   A. Find the marginal density of X

   B. Find the expectation of X, if it exists.

   C. Find the median of X (i.e. the value q such that $P(X \leq q^*) = 0.5$

2. Let X be a random variable with exponential distribution such that $EX = \lambda^{-1}$.

   A. Show that $P(X > s + t|X > s)$ does not depend on s

   B. Find $E(X|X > s)$, for $s > 0$.

   C. Find $E(X|X \leq s)$, for $s > 0$.

3. Let (X, Y) be a bivariate continuous random variable with density

$$f(x, y) = \begin{cases} k(x + y) & 0 \leq x \leq 2; 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

   A. Find the constant k.

   B. Find the covariance between X and Y, $Cov(X, Y)$

   C. Find the density of the random variable $Z = XY$

4. Eye color is determined by a single pair of genes. If both genes received from the parents are those of light-eyes (C) then the child will have light eyes. If both genes are dark-eye (S) genes or the child receives one type C and one type S, then the child will have dark eyes. Each newborn receives a gene from each parent independently and each gene is randomly chosen between the two possessed by each parent. Giacomo has dark eyes (S), and both of his parents have dark eyes but Giacomo's sister has light eyes (C).

   A. What is the probability the Giacomo has a gene of type C?

   Suppose that Giacomo is married with a woman with light eyes

   B. What is the probability that the first child has light eyes?

C. If the first child has dark eyes, what is the probability that the second child has light eyes?

5. Each morning a student takes one of the three books (labelled 1, 2, 3) he owns from his shelf. The probability that he chooses the book with label i is α_i (where $0 < \alpha_i < 1$, i = 1, 2, 3), and choices on successive days are independent. In the evening he replaces the book at the left-hand end of the shelf. Let X_n be the order of the books at the end of day n.

   A. Find the transition matrix of the chain $\{X_n; n = 1, 2 \ldots, \}$

   B. Suppose that at time 0 the books are in the order 1,2,3, from left to right. What is the probability that at the end of day 2 the books are in the same order

   C. If $p_{123}^{(n)}$ denotes the probability that on day n the student finds the books in the order 1,2,3, show that, irrespective of the initial arrangement of the books, $p_{123}^{(n)}$ converges as $n \to \infty$, and determine the limit.
