Stochastic Processes

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Name and surname:

Solve 2 exercises: time 2 hours.

1. An istructor wants to create an exam consisting of 5 problems and covering the 6 sections of the whole course program. To this end, he first make up 10 problems for each of the 6 sections, and then selects at random 5 different problems from these 60 problems

   a) What is the probability that the problems on the exam are all from different program sections

   b) What is the probability that the problems on the exam are all from the same section?

   c) What is the expected number of sections from which there is a problem on the exam?

2. Suppose that the joint density of (X, Y) is

$$f_{XY}(x,y) = \begin{cases} \dfrac{k}{(x\,y)^4} & x > 1, y > 1 \\ 0 & otherwise \end{cases}$$

   a) Find the value of k, the density $f_X(x)$, $E(X)$, $Var(X)$, $Cov(X, Y)$

   b) Let $Z = \max\{X, Y\}$ denote the maximum of the two random variables. Find the density of Z

   c) Consider

$$U = \frac{X}{Y}.$$

   Find $P(U > u)$ for $u > 1$ and $P(U < u)$ for $u \in (0, 1)$. Find the density of U

3. Suppose that X and Y are two independent standard normal random variables. Consider also $U = X/Y$ and $V = X$

   (a) Find $E(|X|)$

   (b) Find the joint density of (U, V)

   (c) Find the marginal density of U

   (d) Find the marginal density of $V|U = u$

   (e) If exists, find $E(U)$

4. Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage (time 0), a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn. Let X_n be the Markov chain indicating the color of the urn extracted at time n

a) Find the transition matrix of the chain X_n

b) Find the probability that the second extraction is done from the red urn

c) Find the probability that at the second extraction you have a white ball.

d) Find the invariant distribution of the chain.

e) If the chain converges, in the long run, what is the probability to extract a white ball?