Ex 1

$|\Omega| = \binom{60}{5}$

a)

D = "all the problems are from different program section"

$$P(D) = \frac{10^5 \binom{6}{5}}{\binom{60}{5}} = \frac{6 \times 10^5}{\frac{60 \times 59 \times 58 \times 57 \times 56}{5!}} = \frac{6!\ 10^5}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56} \approx 0.109$$

b) S = "all from the same section"

$$P(S) = \frac{\binom{6}{1}\binom{10}{5}}{\binom{60}{5}} = \frac{6 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56} = 0.00028$$

c)

$$I_j = \begin{cases} 1 & \text{section } j \text{ is in the exam} \\ 0 & \text{otherwise} \end{cases}$$

$X = \sum_{j=1}^{6} I_j$ = number of sections in the exam

$$E(X) = \sum_{j=1}^{6} E(I_j)$$

$P(I_j = 1) = 1 - P(I_j = 0) = 1 - P(\text{section } j \text{ not at the exam})$

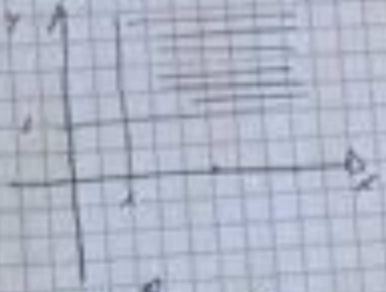
$$= 1 - \frac{\binom{50}{5}}{\binom{60}{5}} = 1 - \frac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56} \approx 0.612$$

$E(X) = 6 \cdot 0.612 = 3.672$

Ex 2

a) $1 = k\int_1^\infty \frac{1}{x^4}\int_1^\infty \frac{1}{y^4}\, dy\, dx = k\int_1^\infty \frac{1}{x^4}\, -\frac{1}{3}\frac{1}{y^3}\Big|_1^\infty dx = k\int_1^\infty \frac{1}{3}\frac{1}{x^4}\, dx = k\cdot -\frac{1}{9}\frac{1}{x^3}\Big|_1^\infty$

$= \frac{k}{9} \Rightarrow k = 9$



$f_X(x) = \begin{cases} \int_1^\infty \frac{9}{x^4 y^4}\, dy & x>1 \\ 0 & x<1 \end{cases} = \begin{cases} \frac{3}{x^4} & x>1 \\ 0 & \text{otherwise} \end{cases}$

$E(X) = \int_1^\infty x\cdot\frac{3}{x^4}\, dx = 3\int_1^\infty \frac{1}{x^3}\, dx = 3\cdot -\frac{1}{2}x^{-2}\Big|_1^\infty = \frac{3}{2}$

$E(X^2) = \int_1^\infty x^2\frac{3}{x^4}\, dx = 3\int_1^\infty \frac{1}{x^2}\, dx = 3\cdot -\frac{1}{x}\Big|_1^\infty = 3$

$VAR(X) = E(X^2) - E(X)^2 = 3 - \frac{9}{4} = \frac{12-9}{4} = \frac{3}{4}$

$Cov(X,Y) = 0$ X and Y are independent.

b) Note that $F_X(x) = \int_1^x \frac{3}{t^4}\, dt = 3\int_1^x t^{-4}\, dt = 3\cdot -\frac{1}{3}t^{-3}\Big|_1^x$

$= t^{-3}\Big|_x^1 = 1 - x^{-3}$ $x>1$

Then for $z>1$

$F_Z(z) = P(Z\le z) = P(X\le z, Y\le z) = P(X\le z)\,P(Y\le z)$

$= (1-z^{-3})(1-z^{-3}) = 1 - z^{-3} - z^{-3} + z^{-6} = 1 - 2z^{-3} + z^{-6}$

$f_Z(z) = -2\cdot(-3)z^{-4} - 6z^{-7}$

$= 6(z^{-4} - z^{-7})$

c) $u < 1$

$P(U \le u) = P\left(\frac{X}{Y} \le u\right) = P\left(\frac{1}{u}X \le Y\right) = P\left(Y \ge \frac{1}{u}X\right) =$

$= \int_1^\infty \int_{\frac{1}{u}x}^\infty \frac{9}{x^4 y^4}\, dy\, dx = \int_1^\infty \frac{9}{x^4}\left(-\frac{1}{3}y^{-3}\right)\Big|_{\frac{1}{u}x}^{\infty} dx =$

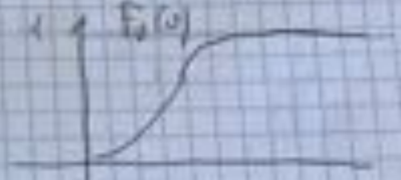
$= \int_1^\infty \frac{9}{x^4}\,\frac{1}{3}\,\frac{u^3}{x^3}\, dx = \int_1^\infty 3u^3 \frac{1}{x^7}\, dx = 3u^3\left(-\frac{1}{6}\frac{1}{x^6}\right)\Big|_1^\infty = \frac{1}{2}u^3 \quad u<1$

note that $\frac{1}{u} < 1$

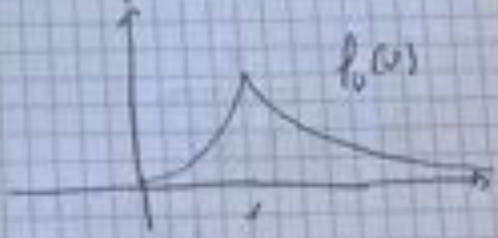
$u > 1$

$P(U > u) = \cancel{P(\ )}\ P\left(\frac{X}{Y} > u\right) = P\left(\frac{1}{u} > \frac{Y}{X}\right) = P\left(\frac{Y}{X} < \frac{1}{u}\right) = \frac{1}{2}\frac{1}{u^3}$

In fact $\frac{Y}{X} = \frac{1}{\frac{X}{Y}}$, that is $\frac{Y}{X}$ and $\frac{X}{Y}$ have the same distribution since X and Y have the same distribution and are independent



Then $F_U(u) = \begin{cases} \frac{1}{2}u^3 & 0<u<1 \\ 1-\frac{1}{2}u^{-3} & 1<u<\infty \end{cases}$



$f_U(u) = \begin{cases} \frac{3}{2}u^2 & 0<u<1 \\ \frac{3}{2}u^{-4} & 1<u<\infty \end{cases}$

Ex 3

$X = V \qquad x = v$

$Y = \frac{X}{U} \qquad y = \frac{v}{u}$

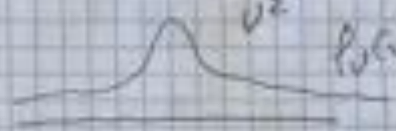
The support of (U,V) is R^2

The Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = -\frac{v}{u^2}$$

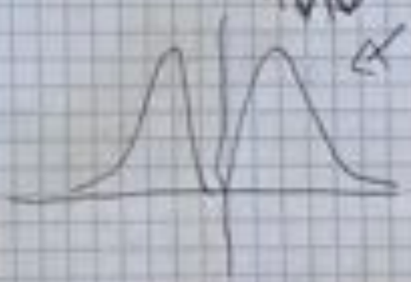
$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{v^2}{u^2}} \frac{|v|}{u^2} \qquad u,v \in R^2$$

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2\left(1+\frac{1}{u^2}\right)} \frac{|v|}{u^2}\, dv$$



$$= \frac{1}{2\pi u^2} 2\int_0^{\infty} e^{-\frac{1}{2}v^2\left(1+\frac{1}{u^2}\right)} v\, dv$$

$$= \frac{1}{\pi u^2} \left. -e^{-\frac{1}{2}v^2\left(1+\frac{1}{u^2}\right)} \frac{1}{\left(1+\frac{1}{u^2}\right)} \right|_{v=0}^{v=\infty}$$



$$= \frac{1}{\pi}\frac{1}{1+u^2} \qquad u \in R$$

$$f_{V|U}(v|u) = \frac{\frac{1}{2\pi} e^{-\frac{1}{2}v^2\left(1+\frac{1}{u^2}\right)} \frac{|v|}{u^2}}{\frac{1}{\pi}\frac{1}{(1+u^2)}} = \frac{1}{2} e^{-\frac{1}{2}v^2\left(1+\frac{1}{u^2}\right)} \left(1+\frac{1}{u^2}\right)|v|$$

$U \sim$ Cauchy $\quad E(U)$ does not exist.

Let X_{n+1} be the urn for the extraction at time $n+1$

Note that X_0 = red. The transition matrix is

	R	W	B
R	1/5	0	4/5
W	2/7	3/7	2/7
B	3/9	4/9	2/9

$P(X_2=R \mid X_0=R) = P(X_2=R, X_1=R \mid X_0=R) + P(X_2=R, X_1=W \mid X_0=R)$

$+ P(X_2=R, X_1=B \mid X_0=R) =$

$= P(X_2=R \mid X_1=R)\, P(X_1=R \mid X_0=R) + P(X_2=R \mid X_1=W)\, P(X_1=W \mid X_0=R)$

$+ P(X_2=R \mid X_1=B)\, P(X_1=B \mid X_0=R) =$

$= \frac{1}{5}\frac{1}{5} + \frac{2}{7}0 + \frac{3}{9}\cdot\frac{4}{5} = \frac{1}{5}\left(\frac{1}{5}+\frac{12}{9}\right) = \frac{1}{5}\frac{9+60}{45} = \frac{69}{225}$

$P(2^{nd}$ extraction a white Ball$) = P(X_3=W \mid X_0=R)$

Now $P(X_2=W \mid X_0=R) = \frac{4}{5} \times \frac{4}{9} = \frac{80}{225}$

$P(X_2=B \mid X_0=R) = \frac{1}{5}\frac{4}{5} + \frac{4}{5}\frac{2}{9} = \frac{1}{5}\left(\frac{4}{5}+\frac{8}{9}\right) = \frac{1}{5}\left(\frac{36+40}{45}\right) = \frac{4}{2}$

$X_2 \mid X_0 \sim \left(\frac{69}{225}\ \ \frac{80}{225}\ \ \frac{46}{225}\right)$ $P(X_3=W \mid X_0=R) = \frac{80}{225}\frac{3}{7} + \frac{46}{225}\frac{4}{9}$

R W B

$$\begin{cases} \pi_R + \pi_W + \pi_B = 1 \\ \pi_R \cdot 0 + \pi_W \frac{5}{7} + \pi_B \frac{2}{9} = \pi_W \\ \pi_R \frac{4}{5} + \pi_W \frac{2}{7} + \pi_B \frac{2}{9} = \pi_B \end{cases}$$

From the second equation

$$\pi_B \frac{2}{9} = \pi_W \frac{2}{7} \iff \pi_W = \frac{7}{9} \pi_B$$

From the third

$$\pi_R \frac{4}{5} + \frac{7}{9}\pi_B \frac{2}{7} + \pi_B \frac{2}{9} = \pi_B \iff \pi_R \frac{4}{5} = \pi_B \left(1 - \frac{2}{9} - \frac{14}{63}\right) = \pi_B \frac{63 - 14 - 14}{63}$$

$$\iff \pi_R = \frac{5}{4} \cdot \frac{35}{63} \pi_B = \frac{25}{36} \pi_B$$

the π $\frac{25}{36}\pi_B + \frac{7}{9}\pi_B + \pi_B = 1 \iff$

$$\pi_B \frac{25 + 28 + 36}{36} = 1$$

$$\pi_B = \frac{36}{89} \qquad \pi_W = \frac{7}{9} \cdot \frac{36}{89} = \frac{28}{89} \qquad \pi_R = \frac{25}{36} \cdot \frac{36}{89} = \frac{25}{89}$$

The chain is irreducible and aperiodic, since it is finite the invariant distribution is also the limit of the chain

Note that $P(\text{white ball at time } n) =$

$$= P(\text{white urn at time } n+1) \xrightarrow[n \to \infty]{} \frac{28}{89}$$

Alternatively note that when $n \to \infty$

$$P(\text{white ball}) = \pi_R \cdot 0 + \pi_W \frac{3}{7} + \pi_B \frac{4}{9} =$$

$$= \frac{\cancel{28}^{4}}{89} \frac{3}{\cancel{7}_1} + \frac{\cancel{36}^{4}}{89} \frac{4}{\cancel{9}_1} = \frac{28}{89}$$