Stochastic Processes

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Name and surname:

Solve 2 exercises: time 2 hours.

1. Let n be the number of students in a class. Suppose that each pair of students can be friends on Facebook with probability μ and that friendships are independent among students. Let D be the number of friendships among the class students

   a) How many possible pairs of friends can we have in the class?

   b) What is the expected number of the random variable D? Which kind of random variable is D?

   c) Now suppose that $n = 10$ and there are 5 girls and 5 boys in the class. Moreover, friendships are only possible between students of the same sex. In this case, what is the expectation of D? Which kind of random variable is D?

   d) Finally, let D_1 the number of friendships among boys and D_2 the number of friendships among girls. Suppose also that $\mu = 1/2$. Find the probability that $D_1 = D_2$

2. Suppose that the joint density of (X, Y) is

$$f_{XY}(x, y) = \begin{cases} c(xy + x + y) & 0 < x < 1, 0 < y < 1 \\ 0 & otherwise \end{cases}$$

   a) Find the value of c, the density $f_X(x)$, $E(X)$, $Cov(X, Y)$

   b) Find $P(Y < X)$. Moreover consider

$$U = \frac{Y}{X}$$

   Find $P(U < u)$ for $u < 1$ and $P(U > u)$ for $u > 1$. Write the distribution function $F_U(u)$

   c) Let $Z = \max\{X, Y\}$ denote the maximum of the two random variables. Find the density of Z

3. Consider a continuous random variable (X, Y) with density

$$f_{XY}(x, y) = \begin{cases} e^{-x} & x > 0, 0 < y < 1 \\ 0 & otherwise \end{cases}$$

   (a) Find the density of Y. Are X and Y independent?

   (b) Let $V = X$ and $Z = X + Y$. Find the density of (V,Z). (Specify the set of values where the density is positive)

   (c) Find the density, the cumulative distribution function, and the expectation of the random variable Z

   (d) Find the covariance between V and Z

4. In a gambling machine, there are two possible strategies, A and B. Suppose that strategy A has a probability of success equal to 1/4, while with strategy B, the probability of success is 1/5. A friend of yours ignores these values and decides to adopt the following rule during a long series of matches. If at time $n-1$ he wins (state S), he does not change strategy at time n. If he loses, (state F) at time n he changes strategy. Consider the Markov chain X_n with states $\{AF, AS, BF, BS\}$ indicating the combination of strategy and bet outcome at time n. Suppose also that at time 0, your friend starts with the strategy A

   a) Find the transition matrix of the chain X_n
   b) Find the probability that at time 2 your friend is in a winning state
   c) Find the invariant distribution of the chain.
   d) Is the chain irreducible and aperiodic?
   e) If the chain converges, in the long run, what is the probability of being in a winning state?
   f) Another friend decides to select each time strategy A with probability 1/2. What is the probability to win the bet in this case?
