Stochastic Processes

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Name and surname:

Solve 2 exercises: time 2 hours.

1. Suppose Xavier has a fair 4-sided die and Yuan has a fair 6-sided die. They roll their dice at the same time (independently) until someone rolls a "1". Let X be the number of rolls of Xavier and Y be the number of rolls of Yuan.

   (a) Find the probabilities $P(X = k)$ and $P(Y = k)$ for $k = 1, \dots,$

   (b) Find the probability that they roll the first 1 at the same time

   (c) Find the probabilities $P(Y < k)$ and $P(Y < X)$

2. Let X and Y have joint density

$$f_{XY}(x, y) = \begin{cases} c(x^2 + y) & 0 < x < y < 2 \\ 0 & otherwise \end{cases}$$

   Compute each of the following

   a) The value of c

   b) The marginal density $f_X(x)$ for all $x \in \mathcal{R}$

   c) The marginal density $f_Y(y)$ for all $y \in \mathcal{R}$

   d) $P(Y < 1)$

   e) $P(X + Y < z)$ for $0 < z < 2$

3. Let X and Y jointly continuous random variable with density

$$f_{XY}(x, y) = \begin{cases} c(4x^2y + 2y^5) & 0 < x < 1, 0 < y < 1 \\ 0 & otherwise \end{cases}$$

   Consider $Z = X + Y^2$ and $W = X - Y^2$

   a) Find the value of c

   b) Write the set where the joint density of (Z, W) is positive

   c) Find the joint density of (Z, W)

   d) Find the marginal density of Z

4. In the two urns A and B there are three red and two green balls. One ball is drawn from the urn containing three balls and it is placed in the other urn. Let X_n be the number of green balls **in the urn that after n draws** contains two balls, for $n = 1, 2, \ldots$. $X_0 = 2$.

   (a) Explain why the sequence $\{X_n; n \geq 0\}$ is a Markov chain

   (b) Find the distribution of X_2

   (c) Find the transition matrix of X_n

   (d) Find the invariant distribution

   (e) Now consider the sequence Y_n where

$$Y_n = \begin{cases} 1 & \text{if the ball drawn in the n:th draw is red} \\ 2 & \text{if the ball drawn in the n:th draw is green} \end{cases}$$

   write the conditional probability $P(Y_n = i | X_{n-1} = j)$ for $i = 1, 2$ and $j = 1, 2, 3$. Is the sequence $\{Y_n; n \geq 0\}$ a Markov chain? Find the distribution of Y_n in the long period?