Stochastic Processes

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Solve 2 exercises: time 2 hours.

1. Suppose Xavier has a fair 4-sided die and Yuan has a fair 6-sided die. They roll their dice at the same time (independently) until someone rolls a "1". Let X be the number of rolls of Xavier and Y be the number of rolls of Yuan.

   (a) Find the probabilities $P(X = k)$ and $P(Y = k)$ for $k = 1, \ldots,$

   $$P(X = k) = \frac{1}{4}\left(\frac{3}{4}\right)^{k-1} \qquad P(Y = k) = \frac{1}{6}\left(\frac{5}{6}\right)^{k-1}$$

   (b) Find the probability that they roll the first 1 at the same time
   Remember that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $|x| < 1$

   $$\begin{aligned} P(Y = X) &= \sum_{k=1}^{\infty} P(Y = k, X = k) = \frac{1}{4}\frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{k-1}\left(\frac{5}{6}\right)^{k-1} \\ &= \frac{1}{24}\sum_{k=1}^{\infty}\left(\frac{15}{24}\right)^{k-1} = \frac{1}{24}\sum_{k=0}^{\infty}\left(\frac{15}{24}\right)^{k} = \frac{1}{24}\frac{1}{1-15/24} = \frac{1}{9} \end{aligned}$$

   (c) Find the probabilities $P(Y < k)$ and $P(Y < X)$

   $$P(Y < k) = 1 - P(Y \geq k) = 1 - \left(\frac{5}{6}\right)^{k-1}$$

   $$\begin{aligned} P(Y < X) &= \sum_{k=1}^{\infty} P(Y < X, X = k) = \sum_{k=2}^{\infty} P(Y < X, X = k) \\ &= \sum_{k=2}^{\infty} P(Y < k, X = k) = \sum_{k=2}^{\infty}\left(1 - \left(\frac{5}{6}\right)^{k-1}\right)\frac{1}{4}\left(\frac{3}{4}\right)^{k-1} \\ &= \left(1 - \frac{1}{4}\right) - \frac{1}{4}\sum_{k=2}^{\infty}\left(\frac{15}{24}\right)^{k-1} = \frac{3}{4} - \frac{1}{4}\frac{15/24}{1-15/24} = \frac{1}{3} \end{aligned}$$

2. Let X and Y have joint density

   $$f_{XY}(x, y) = \begin{cases} c(x^2 + y) & 0 < x < y < 2 \\ 0 & \textit{otherwise} \end{cases}$$

   Compute each of the following

a) The value of c

$$1 = c\int_0^2 \left(\int_x^2 (x^2+y)dy\right) dx = 4$$

c=1/4

b) The marginal density $f_X(x)$ for all $x \in \mathcal{R}$

$$f_X(x) = \int_x^2 \frac{1}{4}(x^2+y)dy = \frac{1}{4}(2+(3/2)x^2 - x^3) \quad x \in (0,2)$$

and 0 otherwise

c) The marginal density $f_Y(y)$ for all $y \in \mathcal{R}$

$$f_Y(y) = \int_0^y \frac{1}{4}(x^2+y)dx = \frac{1}{4}((1/3)y^3 + y^2) \quad y \in (0,2)$$

and 0 otherwise

d) $P(Y < 1)$

$$P(Y<1) = \int_0^1 f_Y(y)dy = \cdots = \frac{5}{48}$$

e) $P(X+Y<z)$ for $0 < z < 2$

The straight lines $y = z - x$ and $y = x$ meet at $x = z/2$. For $0 < z < 2$

$$P(X+Y<z) = \int_0^{z/2}\left(\int_x^{z-x} \frac{1}{4}(x^2+y)dy\right)dx = \cdots = \frac{1}{4}\left(\frac{1}{8}z^3 - \frac{1}{96}z^4\right)$$

3. Let X and Y jointly continuous random variable with density

$$f_{XY}(x,y) = \begin{cases} c(4x^2y+2y^5) & 0<x<1, 0<y<1 \\ 0 & otherwise \end{cases}$$

Consider $Z = X + Y^2$ and $W = X - Y^2$

a) Find the value of c

c=1

b) Write the set where the joint density of (Z,W) is positive

$$A = \{(z,w) : -z < w < z, w \in (0,1)\} \cup \{(z,w) : z-2 < w < 2-z, w \in (1,2)\}$$

c) Find the joint density of (Z,W)

$$f_{ZW}(z,w) = \left(\frac{z+w}{2}\right)^2 + \frac{1}{2}\left(\frac{z-w}{2}\right)^2 = \frac{3}{8}z^2 + \frac{3}{8}w^2 + \frac{1}{2}wz$$

in the set $(z,w) \in A$

d) Find the marginal density of Z

For $z \in (0,1)$

$$f_Z(z) = \int_{-z}^{z} f_{ZW}(z,w)dw = \cdots$$

For $z \in (1,2)$

$$f_Z(z) = \int_{-2+z}^{2-z} f_{ZW}(z,w)dw = \cdots$$

4. In the two urns A and B there are three red and two green balls. One ball is drawn from the urn containing three balls and it is placed in the other urn. Let X_n be the number of green balls **in the urn that after n draws** contains two balls, for $n = 1, 2, \ldots$. $X_0 = 2$.

   (a) Explain why the sequence $\{X_n; n \geq 0\}$ is a Markov chain

   (b) Find the transition matrix of X_n

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 2/3 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

   (c) Find the distribution of X_2

   $\pi_0 = (0, 0, 1)$

$$\pi_1 = \pi_0 P = (1, 0, 0)$$

$$\pi_2 = \pi_1 P = (0, 2/3, 1/3)$$

   (d) Find the invariant distribution

$$\pi = (0.3, 0.6, 0.1)$$

   (e) Now consider the sequence Y_n where

$$Y_n = \begin{cases} 1 & \text{if the ball drawn in the n:th draw is red} \\ 2 & \text{if the ball drawn in the n:th draw is green} \end{cases}$$

   write the conditional probability $P(Y_n = i | X_{n-1} = j)$ for $i = 1, 2$ and $j = 1, 2, 3$. Is the sequence $\{Y_n; n \geq 0\}$ a Markov chain? Find the distribution of Y_n in the long period?

   c) $\{Y_n; n \geq 0$ is not a Markov chain. For example, we have $P(Y_4 = 2 \mid Y_3 = 1, Y_2 = 1, Y_1 = 1) = 2/3$, since if the event $\{Y_3 = 1, Y_2 = 1, Y_1 = 1\}$ occured, there are 1 red and 2 green balls in the urn holding three balls. The probability that you draw a green ball is thus 2/3. On the other hand $P(Y_4 = 2 \mid Y_3 = 1, Y_2 = 2, Y_1 = 1) = 1/3$ since after the third draw there are 2 red and 1 green ball in the urn with 3 balls.
