Stochastic Processes

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Name and surname:

Solve 2 exercises: time 2 hours.

1. Answer the following questions

   (a) How many ways are there to line up six people so that a particular pair of people are adjacent
   $4! \times 5 \times 2$

   (b) How many ways are there to line up six people so that a particular pair of people are not adjacent
   $6! - 4! \times 5 \times 2$

   (c) How many ways are there to line up six people so that three particular pair of people are adjacent
   $(3 \times 2 \times 1) \times 2^3 = 48$

   (d) How many ways are there to line up six people so that none of three particular pair of people are adjacent

2. Let X and Y have joint density

$$f_{XY}(x,y) = \begin{cases} c(x+y) & 0<x<1 \;\; 0<y<1 \\ 0 & otherwise \end{cases}$$

   a) Find the value of c
   1

   b) The marginal density $f_X(x)$ for all $x \in \mathcal{R}$
   $f_X(x) = x + 0.5$ for $x \in (0,1)$

   c) Are X and Y independent?
   No

   d) Find the density of $Z = X + Y$
   For $z \in (0,1)$ $F_Z(z) = z^3/3$, for $z \in (1,2)$ $F_Z(z) = 1 - \frac{1}{3}(4 - 3z^2 + z^3)$

$$F_Z(z) = \begin{cases} \int_0^z \int_0^{z-x} (x+y)dy\,dx = \text{some algebra} = z^3/3 & 0<z<1 \\ 1 - \int_{z-1}^1 \int_{z-x}^1 (x+y)dy\,dx = \text{some algebra} = 1 - (4 - 3z^2 + z^3)/3 & 1<z<2 \end{cases}$$

$$f_Z(z) = \begin{cases} z^2 & 0<z<1 \\ 2z - z^2 & 1<z<2 \end{cases}$$

e) Find the mean and the variance of Z

$$E(z) = \int_0^1 z\, z^2 dz + \int_1^2 z(2z - z^2)dz = 7/6$$

$$E(z^2) = \int_0^1 z^2\, z^2 dz + \int_1^2 z^2(2z - z^2)dz = 3/2$$

$$Var(Z) = 3/2 - 49/36 = 5/36$$

3. Let (X, Y) be a uniform random variable on the triangle $(-1, 0), (0, 1), (1, 0)$

(a) Write the joint density $f_{XY}(x, y)$

$$f_{XY}(x,y) = \begin{cases} 1 & -1 < x < 0,\ y < 1 + x \ \cup\ 0 < x < 1,\ y < 1 - x \\ 0 & otherwise \end{cases}$$

(b) Find the densities $f_X(x)$ and $f_Y(y)$ and the conditional density $f_{Y|X}(y|x)$

$$f_X(x) = \begin{cases} 1 + x & -1 < x < 0 \\ 1 - x & 0 < x < 1 \end{cases}$$

$$f_Y(y) = 2(1 - y) \quad y \in (0, 1)$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{1+x} & 0 < y < 1 + x \quad -1 < x < 0 \\ \frac{1}{1-x} & 0 < y < 1 - x \quad 0 < x < 1 \end{cases}$$

(c) Are X and Y independent No

(d) Let $W = X^2$ find the distribution function and the density of W

For $0 < w < 1$

$$F_W(w) = P(X^2 < w) = P(-\sqrt{w} < X < \sqrt{w}) = 2P(0 < X < \sqrt{w}) = 2(\sqrt{w} - \frac{1}{2}w)$$

$$f_W(w) = w^{-1/2} - 1 \quad w \in (0, 1)$$

(e) let $U = X + Y$ and $V = X - Y$ find the density of (U, V)

$$f_{UV}(u, v) = \frac{1}{2} \quad -1 < u < 1,\ v < u$$

4. A kangaroo jumps between five ordered points $\{A, B, C, D, E\}$ on a circle,. (A is the higher point). At every step he jumps from its location to one of the two neighboring points on the circle with equal probability. Let X_n be the sequence of states occupied by the kangaroo.

(a) Explain why the sequence X_n is a Markov chain

The probability distribution of the position X_n depends only on the state occupied by the kangaroo at time $n-1$

(b) Find the transition matrix of the chain

$$\begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{pmatrix}$$

(c) Suppose that the kangaroo at time 0 is in the point A. Find the distribution of X_2

$$\pi_2 = (1/2, 0, 1/4, 1/4, 0)$$

(d) Is the chain irreducible and aperiodic?

Yes. For the aperiodicity note that $p_{AA}^{(2)} > 0$ and $p_{AA}^{(5)} > 0$ and the g.c.d of 2 and 5 is 1

(e) Find the invariant distribution of the chain

The transition matrix is double stochastic, hence $\pi = (c, c, c, c, c)$ where $c = 1/5$