Stochastic Processes

Prof. Andrea Tancredi

July 2023

Name and surname:

Solve 2 exercises: time 2 hours.

1. Answer the following questions

   (a) How many ways are there to line up six people so that a particular pair of people are adjacent

   (b) How many ways are there to line up six people so that a particular pair of people are not adjacent

   (c) How many ways are there to line up six people so that three particular pair of people are adjacent

   (d) How many ways are there to line up six people so that none of three particular pair of people are adjacent

2. Let X and Y have joint density

$$f_{XY}(x,y) = \begin{cases} c(x+y) & 0<x<1 \;\; 0<y<1 \\ 0 & otherwise \end{cases}$$

   a) Find the value of c

   b) The marginal density $f_X(x)$ for all $x \in \mathcal{R}$

   c) Are X and Y independent?

   d) Find the density of $Z = X + Y$

$$F_Z(z) = \begin{cases} \int_0^z \int_0^{z-x} (x+y)dy\, dx = \text{some algebra} = z^3/3 & 0<z<1 \\ 1 - \int_{z-1}^1 \int_{z-x}^1 (x+y)dy\, dx = \text{some algebra} = 1-(4-3z^2+z^3)/3 & 1<z<2 \end{cases}$$

$$f_Z(z) = \begin{cases} z^2 & 0<z<1 \\ 2z-z^2 & 1<z<2 \end{cases}$$

   e) Find the mean and the variance of Z

3. Let (X,Y) be a uniform random variable on the triangle $(-1,0),(0,1),(1,0)$

   (a) Write the joint density $f_{XY}(x,y)$

   (b) Find the densities $f_X(x)$ and $f_Y(y)$ and the conditional density $f_{Y|X}(y|x)$

   (c) Are X and Y independent

   (d) Let $W = X^2$ find the distribution function and the density of W

(e) let $U = X + Y$ and $V = X - Y$ find the density of (U, V)

4. A kangaroo jumps between five ordered points $\{A, B, C, D, E\}$ on a circle,. (A is the higher point). At every step he jumps from its location to one of the two neighboring points on the circle with equal probability. Let X_n be the sequence of states occupied by the kangaroo.

   (a) Explain why the sequence X_n is a Markov chain

   (b) Find the transition matrix of the chain

   (c) Suppose that the kangaroo at time 0 is in the point A. Find the distribution of X_2

   (d) Is the chain irreducible and aperiodic?

   (e) Find the invariant distribution of the chain
