Stochastic Processes

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Name and surname:

Solve 2 exercises: time 2 hours.

1. An ecologist is collecting butterflies. She collects a total of 50 butterflies. There are a total of 24 species of butterfly, and each butterfly is equally likely to be any of the species

   (a) What is the probability that all the collected butterflies belong to the same species?

   (b) What is the expected number of species of butterflies she collects?

   (c) What is the variance of the number of species of butterflies she collects?

2. Let $X_1, \dots, X_n$ n independent random variables with density function

$$f_X(x) = \begin{cases} 2xe^{-x^2} & x > 0 \\ 0 & otherwise \end{cases}$$

   a) Find the density of $M = \max\{X_1, \dots, X_n\}$ and $U = \min\{X_1, \dots, X_n\}$

   b) Find the joint density of M and U

   c) Find the joint density of $S = M - U, T = M + U$

3. Let (X, Y) be a uniform random variable on the triangle $(-1, 0), (0, 1), (1, 0)$

   (a) Write the joint density $f_{XY}(x, y)$. Are X and Y independent ?

   (b) Find the densities $f_X(x)$ and $f_Y(y)$ and the conditional density $f_{Y|X}(y|x)$

   (c) Let $W = X^2$ find the distribution function and the density of W

   (d) let $U = X + Y$ and $V = X - Y$ find the density of (U, V)

4. A kangaroo jumps between five ordered points $\{A, B, C, D, E\}$ on a circle, (A is the higher point). At every step he moves accordingly to the following strategy: he extracts a card from a desk of french cards. If the card is red (hearths or diamonds) he remains at the same point otherwise he jumps from its location to one of the two neighboring points on the circle with equal probability. Let X_n be the sequence of states occupied by the kangaroo.

   (a) Explain why the sequence X_n is a Markov chain

   (b) Find the transition matrix of the chain

   (c) Suppose that the kangaroo at time 0 is in the point A. Find the distribution of X_2

   (d) Is the chain irreducible and aperiodic?

   (e) Find the invariant distribution of the chain

   (f) What is the probability that at time n the kangaroo visits all the 5 states of the circle in a row

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