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# IN A SIMPLE MACROECONOMIC CONTEXT

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# OPTIMAL FISCAL POLICY IN A SIMPLE MACROECONOMIC CONTEXT

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ABSTRACT

*This article derives optimal fiscal rules within a stochastic model of Keynesian type in the context of Poole (1970) analysis. By using optimal control theory and applying the Hamilton-Jacoby-Bellman equation, we extend the original Poole results concerning the output stabilization properties of monetary policy to the case of fiscal policy. In particular, we look for the optimal setting of government expenditure and lump-sum taxation in the case that the fiscal authority wishes to keep the product close to a reference value and that the economy is assumed to be affected by stochastic disturbances of real and/or monetary type. According to the findings an optimal government expenditure rule is on average preferable to a taxation rule whatever the source of disturbances.*

***Classification JEL***: *C6, E6*.
***Keywords***: *Fiscal Policy, Poole model, Hamilton-Jacoby-Bellman equation*.

## 1. INTRODUCTION

In this paper we address a common question in economics, namely, how should a Ramsey-type authority conduct output stabilization policy. In particular, we analyze how the policy-maker can choose between public expenditure or lump-sum taxation as its optimal policy instrument in a Poole-type model (Poole, 1970). In his pioneering study of the choice of the optimal monetary instrument, Poole has shown that the stochastic structure of the economy and the source of different disturbances would affect the choice of the optimal policy instrument. In particular, he compares a money supply rule to an interest rate rule and uses output

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variability as the sole evaluation criterion. He finds out that an interest rate rule is better than a money supply rule if shocks originate in money markets, whereas a money supply rule gets better results in a context where output shocks operate. The original analysis of Poole was conducted within a stochastic variant of the standard textbook IS-LM model and has exerted a significant influence on both the theoretical literature and the real-world monetary practice.

A natural question stemming from Poole basic insight is whether a similar problem arises in fiscal policy. In this case it might be possible to evaluate the effectiveness of fiscal policy in stabilizating output in a stochastic context.[a]

Interestingly, the extension of the original Poole contribution to fiscal policy remained relatively unexplored. In fact, no existing analyses from that contribution onwards explicitly take into account the relative effectiveness of fiscal instruments in the original Poole context: subsequent developments somewhat broadened the range of issues that could be studied with this model, rather by enriching it with other elements (e.g, the supply side and the rational expectations), notably by incorporating the Poole insights within modern general equilibrium models.

But while Poole's analysis has been generalized in several directions, e.g. to include a variable price level, an output/output gap distinction, and interest rate feedback rules, these extensions mainly involve changes to the aggregate-supply and policy-rule specifications, so the advent of the optimizing IS-LM equations do not get a critical difference to Poole's results.

The major extension of the basic Poole model is by Canzoneri et al.(1983), who augment the closed-economy Poole analysis with rational expectations and imperfect information. They show that the results of the original Poole analysis with respect to liquidity as well as output demand shocks are exactly replicated. Recently, in a standard New-Keynesian model Collard and Dellas (2005) examine the properties of alternative targeting procedures in an economy that represents a general equilibrium rendition of Poole contribution. In their model, they obtain the original Poole results concerning output stabilization properties of money and interest rate when the degree of intertemporal substitution is 'sufficiently' low (see also Collard and Dellas 2000). Hoffmann and Kempa (2009) extend the original Poole analysis to a general two country open-economy context for a large economy and a small open economy. Their results for the large economy resemble those of the original Poole analysis whereas in the small economy scenario the results of the large economy continue to hold only for domestic shocks.

The present paper instead takes an alternative line with respect the current literature.

[a]In fact, in the conclusion of his paper Poole (1970) states that: "*While the instrument problem has been analyzed as a monetary policy problem, it is worth pointing out that a similar problem arises in fiscal policy".*

In fact, our discussion of fiscal policy effectiveness is based on the original IS-LM model where the intertemporal optimization and expectations do not play any role in accounting for aggregate economic activity. In fact, we deal with this issue in a context of the original Poole stochastic model as we extend the original results concerning the output stabilization properties of monetary policy to the case of fiscal policy. A central feature of our analysis is that optimal policy is derived in a highly stylized environment. A key advantage of this stylized approach is that it facilitates understanding the ways through which policy should respond to stabilize output as a consequence of a particular shock in isolation. Furthermore, the mathematical technique is enough general and flexible to take into account the nature of the shock both of real and monetary type.

In fact, the mathematical tool we use is the Hamilton-Jacoby-Bellman equation applied to the minimization of an integral operator whose variable is described by a controlled stochastic differential equation. This procedure is independent from the nature of the shock.

Finally, while policy results are generally drawn with numerical simulations we developed a mathematical technique for the effects to be computed analytically with an explicit solution that makes more transparent the mechanism at work.

We find two main results. First, the source of economy disturbances affects the effectiveness of fiscal policy. Second, an optimal expenditure rule is always preferable to a taxation rule.

The remainder of the paper is organized as follows. Section 2 is devoted to a brief description of the model setup and discusses how real and monetary uncertainty works. Section 3 describes how fiscal authority solves the optimal problem of fiscal policy and explores which instrument, public expenditure or lump-sum taxation, is the optimal one in that context. Section 4 concludes.

## 2. THE SETUP

Following Poole (1970), the problem of a benevolent Ramsey-type authority is to keep the current output near an established reference value denoted by $Y_f$. Moreover, it is assumed to have always access to the available policy instruments, that is lump-sum taxation and government expenditure.

The authority is increasingly worse-off the larger the deviations are from the reference value. Namely, the fact of having a fiscal authority that increasingly dislikes lower or larger output with respect to the reference value is taken into account by assuming a quadratic expected loss function $J$ of the type:

$$J = \mathbf{E}[(Y_T - Y_f)^2], \tag{1}$$

where $[0, T]$ is the time span over which we work, $Y_T$ is the stochastic level of output at time $T$, $Y_f$ is the deterministic reference value and $\mathbf{E}$ denotes the expectation with respect to the probability law of $Y$.

The economy, that constitutes the constraint for the authority, is represented by the traditional IS-LM model properly augmented to include stochastic shocks of fiscal policy.

Thus, the authority wishes to minimize this function $(1)$ subject to the constraint imposed by a stochastic fixed-price IS-LM model in differential form[b]:

$$dY_t = dC_t + dI_t + dG_t, \tag{2}$$

$$dC_t = c dY_t^D, \tag{3}$$

$$dY_t^D = dY_t - dT_t, \tag{4}$$

$$dI_t = \gamma dY_t^D - h dr_t + \lambda_1 u_t dB_t, \tag{5}$$

$$dM_t - dP_t = l_1 dY_t^D - l_2 dr_t + \lambda_2 u_t dB_t, \tag{6}$$

where the symbols are the usual ones and the equations are self-explanatory: in the order, they are the resource constraint of the economy (the ex-ante equality between investment and saving), consumption equation $C_t$ and the investment equation $I_t$ that are function of national disposable income $Y_t^D$ (and the interest rate $r_t$ for the investment), the demand for money equation (as a function of national disposable income and the interest rate as well), $G_t$ represents the government expenditure; $T_t$ is the lump-sum taxation; $0 < c < 1$ and $\gamma > 0$, represent the speed of adjustment in the goods market; $h > 0$, $l_1 > 0$, $l_2 > 0$ the speed of adjustment in the money market. In the following, we also assume that money supply and prices do not change $(dM_t = 0$ and $dP_t = 0)$. Moreover, $u_t$ represents the derivatives of the available instrument (public expenditure or taxation) with respect to time.

Furthermore, $dB_t$ is a Brownian motion and actually the source of the system uncertainty.

It is worth remembering that a diffusion process is a continuous version of the random walk, which is a solution of a stochastic differential equation. It is a continuous-time Markov process with continuous sample paths.

In the case that the uncertainty is not present, the coefficient of $dB_t$ in (5) and in (6) is null. In this case the model collapses to $dY_t^* = \mu_i u_t^{(i)}$ where the control variable $u_t^{(i)}$ represents the variation rate of the control variable and accordingly,

[b]We get the model in differential form to take properly into account the uncertainty. In fact, in the case of the textbook formalization of IS-LM model, the fiscal uncertainty actually would not operate and, as consequence, the fiscal instruments would be basically equivalent in stabilizing output.

$\mu_i$ is Keynesian multiplier associated to the fiscal instrument. The index $i$ runs over $G, T$ and in particular it is set to $G$ if the instrument is the public expenditure, whereas it is $T$ if the instrument is the taxation policy.

The keynesian multiplier will be

$$\mu_G = \frac{1}{1 - c - \gamma + hl_1/l_2} \tag{7}$$

if the control variable is the public expenditure, or

$$\mu_T = \frac{-c - \gamma + hl_1/l_2}{1 - c - \gamma + hl_1/l_2} \tag{8}$$

when the control variable is taxation (see the appendix for details).

According to the stability conditions of the system, taxation multiplier must be negative, so that the following relationship must hold jointly:

$$-c - \gamma + \frac{hl_1}{l_2} < 0 \tag{9}$$

and

$$1 - c - \gamma + \frac{hl_1}{l_2} > 0\,. \tag{10}$$

Both these two conditions boil down to the following relationship:

$$0 < c + \gamma - \frac{hl_1}{l_2} < 1. \tag{11}$$

In the following analysis, we will discuss the consequences of allowing for uncertainty, alternatively, from real or monetary market involving different effects in term of fiscal instrument effectiveness.

The first case corresponds essentially to the IS curve shifts due to stochastic disturbance if the real interest rate was not to change. The second case is when monetary disturbances make it impossible to fix the shape of the LM curve in order to stabilize the system.

Then, we extend our analysis to the case of disturbances of both real and monetary side.

In order to illustrate this point according to the source of stochastic disturbances, it is necessary to adopt a flexible and general approach which may be able to easily describe both cases. The description in this section is based on a general representation of the cases at work. In fact, the model here is flexible enough to allow for a joint representation of a IS-LM with stochastic disturbances in both real and monetary markets.

In fact, through simple calculus, it is possible to determine the reduced form of the model described above in the case that equation $(5)$ is stochastic and equation $(6)$ is deterministic ($\lambda_2 = 0$, the case that uncertainty stems from real market only: see appendix A for more details)[c]:

$$
\begin{aligned}
&dY_t = \mu_i \, u_t^{(i)} \, dt + \sigma \, u_t^{(i)} \, dB_t, \\
&Y_0 = y,
\end{aligned}
\tag{12}
$$

where $\mu_i$ is the Keynesian multiplier associated to the policy instrument considered (taxation $T$ or public expenditure $G$), $u_t$ represents the derivatives of the instrument with respect to time and $\sigma$ (with $\sigma \in \mathbb{R}$) denotes the diffusion coefficient that can assume different functional forms according to the nature of the disturbance (real disturbance, monetary disturbance or both). In this case of real disturbances we have

$$
\sigma = \frac{\lambda_1}{1 - c - \gamma + hl_1/l_2}. \tag{13}
$$

In the sequel, in order to simplify the mathematical notation, we set: $u_t^{(i)} \equiv u_t$ as the two indices vary according to the instrument used in the analysis.

Then, it is possible to determine the reduced form of the model described above in the case that equation $(6)$ is stochastic and equation $(5)$ is deterministic ($\lambda_1 = 0$: the case that uncertainty stems from monetary market only: see appendix A for more details):

$$
\begin{aligned}
&dY_t = \mu_i \, u_t \, dt + \chi \, u_t \, dB_t, \\
&Y_0 = y.
\end{aligned}
\tag{14}
$$

In this case with monetary shock, the diffusion coefficient is

$$
\chi = \frac{-h\lambda_2/l_2}{1 - c - \gamma + hl_1/l_2}. \tag{15}
$$

When real and monetary disturbances are simultaneous (i.e. $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ ), the equation describing the economy becomes (see appendix A for more details):

$$
\begin{cases}
dY_t = \mu_i \, u_t \, dt + (\sigma + \chi) \, u_t \, dB_t, \\
Y_0 = y.
\end{cases}
\tag{16}
$$

Now, we shortly illustrate some mathematical details underlying equation (16). These points ought be useful to better clarify the mathematical notation in the text but also to allow for the instruments we are going to use to solve the optimal control problem.

[c]This formulation is a special case of the more general controlled Itô diffusion of the type:

$$
dY_t = \mu(Y_t, u_t) \, dt + \sigma(Y_t, u_t) \, dB_t, \quad \text{with} \quad Y_0 = y \,,
$$

where $\mu(Y_t, u_t) = \mu u_t$ and $\sigma(Y_t, u_t) = \sigma u_t$.

Since for every $t$ we shall have a random control variable upon which the random variable $Y_t$ depends, we consider a probability space with filtration

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}) \,,$$

where the filtration $\mathcal{F}_t$ is the one generated by the standard 1-dimensional Brownian motion $B$ and is augmented by $\mathbb{P}$-null sets, that is:

$$\mathcal{F}_t = \sigma\Big(B_s,\ \ 0 \leq s \leq t\Big) \cup \Big\{A \in \mathcal{F} \,|\, \mathbb{P}(A) = 0\Big\}, \qquad \forall\ t \geq 0. \tag{17}$$

Moreover, we suppose that $Y_0$ is an integrable random variable with law $\pi_0$ and measurable with respect to $\mathcal{F}_0$ representing the initial value of the current income: for sake of simplicity, we assume that at the time $0$ income is deterministic so that: $Y_0 = y$, constant with $y > 0$.

The control variable $u_t = u(t, \omega)$ is taken in a given family $\mathcal{A}$ of *admissible* controls:

$$\begin{aligned} \mathcal{A} := \Big\{ u_t = u_0(t, Y_t(\omega)) \text{ for some} \\ \text{Borel-measurable functions } u^0 : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R} \Big\} \ . \end{aligned} \tag{18}$$

For more technical details about (17) and (18) see Appendix C.

The value at time $t$ of this functions only depends on the state of the system at this time. These are called *Markov controls* because with such a $u$ the corresponding process $Y_t$ becomes an Itô diffusion, in particular a Markov process.

We are now able to minimize the operator $J$ given in (1) under the constraint (12) or, alternatevely, $(14)$ or $(16)$ involved by the economy. To this aim, we use the Hamilton-Jacobi-Bellman (HJB) equation (see theorem 2 in the appendix B).

As usual in the dynamic programming literature we now let the controlled diffusion $Y$ start at time $s$ from level $y > 0$; that is, we write

$$\begin{cases} dY_t^{s,y} = \mu_i \, u_{t-s} \, dt + \tilde{\sigma} \, u_{t-s} \, dB_{t-s} \,, \quad \text{with} \quad s \leq t \leq T \\ \\ \text{sub} \qquad Y_s^{s,y} = y \,. \end{cases} \tag{19}$$

and the optimization problem now reads

$$\phi(s,y) = \inf_{u \in \mathcal{A}} \mathbf{E}\left[(Y_T^{s,y} - Y_f)^2\right] \ . \tag{20}$$

To apply the HJB method, we have preliminarly to transform the mean value $J$, given in the (1), into the mean value of an integral by Dynkin's formula (see theorem 1 of the appendix B).

As a consequence of Dynkin's formula the mean value $J$, given in (1), reads:

$$J(s,y;u) = (y - Y_f)^2 + \mathbf{E}\left[\int_s^T [2\,\mu_i\, u_{t-s}\,(Y_t^{s,y} - Y_f) + \sigma^2 u_{t-s}^2]\, dt\right] \tag{21}$$

and by virtue of the invariance of the problem under time translation (that is, since the problem is time homogeneous), we can rewrite $J$ in the form

$$J(s,y;u) = (y-Y_f)^2 + \mathbf{E}\left[\int_0^{T-s} [2\,\mu_i\,u_t\,(Y_t - Y_f) + \sigma^2 u_t^2]\,dt\right], \qquad (22)$$

and the optimization problem can be written in the form:

$$\begin{cases} \phi(s,y) := (y-Y_f)^2 + \inf_{u\in\mathcal{A}} \mathbf{E}\left[\int_0^{T-s} [2\,\mu_i\,u_t\,(Y_t - Y_f) + \sigma^2 u_t^2]\,dt\right] \\ \text{sub} \qquad dY_t = \mu_i\,u_t\,dt + \tilde\sigma\,u_t\,dB_t, \qquad \text{with} \quad Y_0 = y\,. \end{cases} \qquad (23)$$

where $\tilde\sigma$ denotes a suitable value of $\sigma$, i.e. $\chi$ or $\sigma+\chi$ and we have simply denoted by $Y_t$ the process $(Y_t^{0,y})_{t\le T}$.

## 3. SOLVING THE OPTIMAL CONTROL PROBLEM FOR THE AUTHORITY

If we apply the HJB equation to the second term on the left hand side of the optimization problem in (23) we obtain (for more detalails see appendix B):

$$\inf_{u\in\mathcal{A}} \left\{ \left[2\,\mu_i u\,(y-Y_f) + \tilde\sigma^2 u^2\right] + \left[\frac{\partial\phi}{\partial s} + \mu_i u \frac{\partial\phi}{\partial y} + \left(\frac{\tilde\sigma^2 u^2}{2}\right) \frac{\partial^2\phi}{\partial y^2}\right] \right\} = 0\,. \qquad (24)$$

To find an optimal control, we now derive equation (24) with respect to $u$ and obtain

$$u = \bar u = \frac{-2\mu_i(y-Y_f) - \mu_i \dfrac{\partial\phi}{\partial y}}{2\tilde\sigma^2 + \tilde\sigma^2 \dfrac{\partial^2\phi}{\partial y^2}}. \qquad (25)$$

Substituting (25) into (24) we then obtain the value function:

$$2\,\mu_i\bar u\,(y-Y_f) + \tilde\sigma^2\bar u^2 + \frac{\partial\phi}{\partial s} + \mu_i\bar u\frac{\partial\phi}{\partial y} + \left(\frac{\tilde\sigma^2\bar u^2}{2}\right)\frac{\partial^2\phi}{\partial y^2} = 0. \qquad (26)$$

Let us try to find a solution $\phi(s,y)$ of (25) and (26) of the form

$$\phi(s,y) = (y-Y_f)^2\; g(s), \qquad (27)$$

with $g(T)=1$.

By substitution of (27) into (25), we obtain:

$$\bar u(s,y) = \frac{-\mu_i\,(y-Y_f)}{\tilde\sigma^2}, \qquad (28)$$

that is the optimal Markov control

$$u(s,\omega) = \bar{u}(s, Y_s(\omega)) = \frac{-\mu_i\,(Y_s - Y_f)}{\tilde{\sigma}^2}\,, \tag{29}$$

which, as we can easily observe, belongs to the set $\mathcal{A}$ given in the (18).

Substituting (27) into (26), we obtain:

$$g'(s) - \frac{\mu_i^2}{\sigma^2}\, g(s) = \frac{\mu_i^2}{\tilde{\sigma}^2}, \tag{30}$$

with $g(T) = 1$, that is

$$g(s) = 2\,\exp\left[\frac{\mu_i^2}{\tilde{\sigma}^2}\,(s - T)\right] - 1. \tag{31}$$

By substitution of (31) into (27) and then into (23), we finally obtain the value function

$$\phi(s,y) = 2\,(y - Y_f)^2\,\exp\left[\frac{\mu_i^2}{\tilde{\sigma}^2}\,(s - T)\right]. \tag{32}$$

If we substitute the solution $u$ given in (29) into the dynamic (16), we obtain the linear geometric stochastic differential equation:

$$d\tilde{Y}_t = -\frac{\mu^2}{\tilde{\sigma}^2}\,\tilde{Y}_t - \frac{\mu}{\tilde{\sigma}}\,\tilde{Y}_t\,dB_t\,, \tag{33}$$

where we have set $\tilde{Y}_t = |Y_t - Y_f|$. By Îto's formula the solution of equation (33) is

$$\tilde{Y}_t = \tilde{Y}_0\,e^{-\mu^2 t/(2\tilde{\sigma}^2)} e^{-\mu B_t/\tilde{\sigma}} \tag{34}$$

where $\tilde{Y}_0 > 0$ and the mean value of the random variable $\tilde{Y}_t$ is

$$\begin{aligned} \mathbf{E}[Y_t - Y_f] = \tilde{Y}_0\,e^{-\mu^2 t/(2\tilde{\sigma}^2)}\mathbf{E}\left[e^{-\mu B_t/\tilde{\sigma}}\right] = \\ = \tilde{Y}_0\,e^{-\mu^2 t/(2\tilde{\sigma}^2)}\,\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-x^2/(2t) - \mu x/\tilde{\sigma}}\,dx = \tilde{Y}_0\,, \end{aligned} \tag{35}$$

because the probability law of $\tilde{Y}_t$ is the one of the Brownian motion $B_t$ whose density function is

$$f(x) = \frac{1}{\sqrt{2\pi t}}\,e^{-x^2/(2t)}\,. \tag{36}$$

If we rewrite (34) in the form

$$\log\tilde{Y}_t = \log\tilde{Y}_0 - \frac{\mu^2 t}{2\sigma^2} - \frac{\mu B_t}{\sigma}\,, \tag{37}$$

where the last term only is stochastic and the first two are deterministic, we can obtain the mean value

$$\mathbf{E}[\log \tilde{Y}_t] = \log \tilde{Y}_0 - \frac{\mu^2 t}{2\sigma^2}\,. \tag{38}$$

In order to compare terms we can take into account that the function *logarithm* is increasing and then does not change the inequality relation between its arguments.

According to the instrument we can substitute for the IS-LM parameters in place of $\mu$ and $\tilde{\sigma}$. We can thus rewrite the optimal problem solution as

$$\mathbf{E}\left[\log \tilde{Y}_t\right] = \log \tilde{Y}_0 - \frac{t}{2\lambda_1^2}, \tag{39}$$

if the policy instrument is public expenditure and:

$$\mathbf{E}\left[\log \tilde{Y}_t\right] = \log \tilde{Y}_0 - \frac{t\left(-c-\gamma+hl_1/l_2\right)^2}{2\lambda_1^2}, \tag{40}$$

if the policy instrument is taxation.

By following the same procedure we have the solution in the case that uncertainty is associated to the money market (i.e., $\lambda_1 = 0$):

$$\mathbf{E}\left[\log \tilde{Y}_t\right] = \log \tilde{Y}_0 - \frac{\mu_i^2\, t}{2\chi^2}\,. \tag{41}$$

Accordingly, if public expenditure is the policy variable equation $(41)$ becomes (see notation reported in appendix A for more details):

$$\mathbf{E}\left[\log \tilde{Y}_t\right] = \log \tilde{Y}_0 - \frac{t}{2\left(h\lambda_2/l_2\right)^2}, \tag{42}$$

while if policy instrument is taxation $(41)$ becomes:

$$\mathbf{E}\left[\log \tilde{Y}_t\right] = \log \tilde{Y}_0 - \frac{t\left(-c-\gamma+hl_1/l_2\right)^2}{2\left(h\lambda_2/l_2\right)^2}. \tag{43}$$

When shocks are simultaneous, the solution is:

$$\mathbf{E}\left[\log \tilde{Y}_t\right] = \log \tilde{Y}_0 - \frac{\mu_i^2\, t}{2\left(\chi+\tilde{\sigma}\right)^2}, \tag{44}$$

that after substitution becomes:

$$\mathbf{E}\left[\log \tilde{Y}_t\right] = \log \tilde{Y}_0 - \frac{t}{2\left(\lambda_1 - h\lambda_2/l_2\right)^2}, \tag{45}$$

when policy instrument is public expenditure, and:

$$\mathbf{E}\left[\log \tilde{Y}_t\right] = \log \tilde{Y}_0 - \frac{t\left(-c-\gamma+hl_1/l_2\right)^2}{2\left(\lambda_1 - h\lambda_2/l_2\right)^2}, \tag{46}$$

if the policy-maker opts for taxation.

Equations (38), (41) and (44) are the key findings of the paper. They describe the optimal path of income gap generated by the policy instrument adopted and affected by the stochastic disturbance.

We can now answer the question whether it is public expenditure or lump-sum taxation the most effective instrument to stabilize output if stochastic shocks operate. The result can be summarized by the following proposition:

**Proposition 1:** *Public expenditure is on average more effective in stabilizing output than lump-sum taxation. Interestingly, this holds whatever the nature of the shock (if it is of real or monetary type or both).*

*Proof:* The proof is based on the following lines. First we draw a comparison between the two policy instruments to address which of them is better than other in the case the uncertainty comes alternatevely, from real or monetary side. Then we analyse which instrument fares better in the case that both types of disturbances are at work.

By comparing the different specifications of income gap in (39) and (40) (the disturbance is only on the real side) we obtain that public expenditure is more effective than taxation if:

$$-\frac{t}{2\lambda_1^2} < -\frac{\left(-c-\gamma+hl_1/l_2\right)^2 t}{2\lambda_1^2}. \tag{47}$$

By the same criterion, comparison between (42) and (43) (the shock stems from monetary market only) shows that public expentiture is more effective than taxation if:

$$-\frac{t}{2\left(h\lambda_2/l_2\right)^2} < -\frac{\left(-c-\gamma+hl_1/l_2\right)^2 t}{2\left(h\lambda_2/l_2\right)^2}. \tag{48}$$

Finally, by comparing (45) and (46) (disturbances are both of real and monetary type), public expenditure is more effective than taxation if:

$$-\frac{t}{2\left(\lambda_1 - h\lambda_2/l_2\right)^2} < -\frac{\left(-c-\gamma+hl_1/l_2\right)^2 t}{2\left(\lambda_1 - h\lambda_2/l_2\right)^2}. \tag{49}$$

Conditions (47), (48) and (49) are basically equivalent and boil down to the following inequality:

$$-1 < c+\gamma-\frac{hl_1}{l_2} < 1\,. \tag{50}$$

To sum up, if (50) holds then public expenditure is preferable to taxation. Interestingly, this condition always must hold as we assume the model stability condition (11).

Thus, condition (50) involves that under the usual assumptions (see equation (11)) public expenditure is on average the most effective instrument to stabilize the product near the reference value. □

## 4. Conclusions

This paper addresses the optimal stabilization policy in the Poole original context. We find that the effectiveness of fiscal policy is affected by the nature of economy disturbances whatever the instrument used. Furthermore, an optimal government expenditure rule is preferable to a taxation rule under the usual stability conditions. Plainly, this model is too simple to be taken seriously as the basis policy evaluation. Nonetheless, the simple analysis here may be both of pedagogical value and useful to pin down the determinants of the product in the well-known (stochastic) IS-LM context. The paper could also be taken as an example of integration between stochastic process optimal control methods and continuous time stochastic models. We developed a methodology which generates stochastic processes as closed form solutions. Accordingly, it is possible to establish some relationships between the structural parameters of the economy and the optimal fiscal rules. A promising development of this model might be the introduction of distortionary taxation, coupled with the extension to a more complicated environment with lagged responses to the disturbances and policy actions.

## 5. Appendix A

In this Appendix we basically develop the same model in the text by explicitly unveiling how the model parameters enter into the reduced form, according to the instrument and the source of uncertainty.

A more compact representation of it is not allowed in this case as a consequence of the flexible approach in use.

### 5.1 Real and Monetary Uncertainty: The case of public expenditure

$$dY_t = dC_t + dI_t + dG_t, \tag{51}$$

$$dC_t = c\, dY_t^D, \tag{52}$$

$$dY_t^D = dY_t - dT_t, \tag{53}$$

$$dI_t = \gamma Y_t^D - h\,dr_t + \lambda_1\,u_t\,dB_t, \tag{54}$$

$$l_1 Y_t^D - l_2\,dr_t + \lambda_2\,u_t\,dB_t = 0, \tag{55}$$

where $dB_t$ is a Brownian motion and, in this case, the source of the system uncertainty and $u_t$ is the variation of the control variable.

Let us solve equation (55) in $dr_t$:

$$dr_t = \frac{\lambda_2}{l_2}\,u_t\,dB_t + \frac{l_1}{l_2}\,Y_t^D. \tag{56}$$

Then, by substituting it in the equation (54) and assuming that the control variable is the public expenditure $u_t = dG_t/dt$ we have:

$$dI_t = \frac{dG_t}{dt}\,dB_t\left(\lambda_1 - \frac{h\lambda_2}{l_2}\right) + Y_t^D\left(\gamma - \frac{hl_1}{l_2}\right). \tag{57}$$

Now we substitute the equations (57), (52) and (53) in (51):

$$dY_t = cdY_t^D + \frac{dG_t}{dt}\,dB_t\left(\lambda_1 - \frac{h\lambda_2}{l_2}\right) + dY_t^D\left(\gamma - \frac{hl_1}{l_2}\right) + dG_t, \tag{58}$$

and, given that $dT_t = 0$ we have:

$$dY_t\left(1 - c - \gamma + \frac{hl_1}{l_2}\right) = u_t\,dB_t + \left(\lambda_1 - \frac{h\lambda_2}{l_2}\right) + dG_t\,, \tag{59}$$

that can be written down in a more compact form:

$$dY_t = \frac{1}{1 - c - \gamma + hl_1/l_2}\,dG_t + \frac{1}{1 - c - \gamma + hl_1/l_2}\left(\lambda_1 - \frac{h\lambda_2}{l_2}\right)\frac{dG_t}{dt}\,dB_t\,. \tag{60}$$

Equation (60) is a particular form of equation (16) with:

$$\mu_i = \mu_G = \frac{1}{1 - c - \gamma + hl_1/l_2}\,, \qquad \sigma + \chi = \frac{\lambda_1 - h\lambda_2/l_2}{1 - c - \gamma + hl_1/l_2} \quad \text{and} \quad u_t = \frac{dG_t}{dt}\,. \tag{61}$$

5.2 Real and Monetary Uncertainty: The case of lump-sum taxation

Now we assume $dG_t = 0$ and $u_t = \frac{dT_t}{dt}$. Repeatig the same procedure we obtain:

$$dY_t = \frac{-c - \gamma + hl_1/l_2}{1 - c - \gamma + hl_1/l_2}\,dT_t + \frac{\lambda_1 - h\lambda_2/l_2}{1 - c - \gamma + hl_1/l_2}\,\frac{dT_t}{dt}\,dB_t\,. \tag{62}$$

Equation $(62)$ is a particular form of equation $(12)$ with:

$$\mu_i = \mu_T = \frac{-c-\gamma+hl_1/l_2}{1-c-\gamma+hl_1/l_2}\,,$$
$$\sigma+\chi = \frac{\lambda_1 - h\lambda_2/l_2}{1-c-\gamma+hl_1/l_2} \qquad \text{and} \qquad u_t = \frac{dT_t}{dt}\,. \tag{63}$$

5.3 Real Uncertainty: The case of public expenditure

Let us assume $u_t = dG_t/dt$ and that uncertainty is on the real side, $\lambda_2 = 0$. Equation $(60)$ becomes:

$$dY_t = \frac{1}{1-c-\gamma+hl_1/l_2}\,dG_t + \frac{\lambda_1}{1-c-\gamma+hl_1/l_2}\,\frac{dG_t}{dt}\,dB_t. \tag{64}$$

Equation $(64)$ is a particular form of equation $(12)$ with:

$$\mu_i = \mu_G = \frac{1}{1-c-\gamma+hl_1/l_2}\,,$$
$$\sigma = \frac{\lambda_1}{1-c-\gamma+hl_1/l_2} \qquad \text{and} \qquad u_t = \frac{dG_t}{dt}\,. \tag{65}$$

5.4 Real Uncertainty: The case of lump-sum taxation

Let us assume $u_t = dT_t/dt$, and that uncertainty is on the real side, $\lambda_2 = 0$. Equation $(62)$ becomes:

$$dY_t = \frac{-c-\gamma+hl_1/l_2}{1-c-\gamma+hl_1/l_2}\,dT_t + \frac{\lambda_1}{1-c-\gamma+hl_1/l_2}\,\frac{dT_t}{dt}\,dB_t\,. \tag{66}$$

Equation $(66)$ is a particular form of equation $(12)$ with

$$\mu_i = \mu_T = \frac{-c-\gamma+hl_1/l_2}{1-c-\gamma+hl_1/l_2}\,,$$
$$\sigma = \frac{\lambda_1}{1-c-\gamma+hl_1/l_2} \qquad \text{and} \qquad u_t = \frac{dT_t}{dt}\,. \tag{67}$$

5.5 Monetary Uncertainty: The case of public expenditure

Let us assume $u_t = dG_t/dt$ and that uncertainty is on the monetary side, $\lambda_1 = 0$. Equation $(60)$ becomes:

$$dY_t = \frac{1}{1-c-\gamma+hl_1l_2}\;dG_t + \frac{-h\lambda_2/l_2}{1-c-\gamma+hl_1/l_2}\;\frac{dG_t}{dt}\;dB_t\,. \tag{68}$$

Equation $(68)$ is a particular form of equation $(14)$ with

$$\begin{aligned} \mu_i = \mu_G = \frac{1}{1-c-\gamma+hl_1/l_2}\,, \\ \chi = \frac{-h\lambda_2/l_2}{1-c-\gamma+hl_1/l_2} \qquad \text{and} \qquad u_t = \frac{dG_t}{dt}\,. \end{aligned} \tag{69}$$

5.6 Monetary Uncertainty: The case of lump-sum taxation

Let us assume $u_t = dT_t/dt$ and that uncertainty is on the monetary side, $\lambda_1 = 0$. Equation $(62)$ becomes:

$$dY_t = \frac{-c-\gamma+hl_1/l_2}{1-c-\gamma+hl_1/l_2}\; dT_t + \frac{-h\lambda_2/l_2}{1-c-\gamma+hl_1/l_2}\; \frac{dT_t}{dt}\; dB_t\,. \tag{70}$$

Equation $(70)$ is a particular form of equation $(14)$ with

$$\begin{aligned} \mu_i = \mu_T = \frac{-c-\gamma+hl_1/l_2}{1-c-\gamma+hl_1/l_2}\,, \\ \chi = \frac{-h\lambda_2/l_2}{1-c-\gamma+hl_1/l_2} \qquad \text{and} \qquad u_t = \frac{dT_t}{dt}\,. \end{aligned} \tag{71}$$

Appendix B

**Remark 2.** *There exists a unique solution for the controlled equation (12) and we refer the reader, for example, to Øksendal (2003) for the proof.*

For $\bar{Y} < Y_f$ let us define the exit time $\tau$ of the dynamics as

$$\tau := \inf\{t > 0 \;\mid\; Y_t > \bar{Y}\}.$$

**Remark 3.** *By virtue of well known results, the measurability of $\tau$ with respect to the $\sigma$-algebra $\mathcal{F}_t$ follows. Indeed, we have that $\tau$ is a stopping time.*

**Theorem 4 (Dynkin's formula).** *Let $Y_t$ be the Itô diffusion*

$$dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dB_t, \quad Y_0 = y,$$

*and $f \in C_0^2(\mathbb{R})$. If $\tau$ is a stopping time with $\mathbf{E}[\tau] < +\infty$, then it yealds*

$$\mathbf{E}[f(Y_\tau)] = f(y) + \mathbf{E}\left[\int_0^\tau Lf(Y_s)\,ds\right], \tag{72}$$

*where*

$$Lf(z) := \mu(z)\,\frac{df}{dz} + \frac{1}{2}\,[\sigma(z)]^2\,\frac{d^2 f}{dz^2}\,. \tag{73}$$

**Theorem 5 (HJB equation).** *Suppose that we have*

$$V(s,y) := \sup_{u_t \in \mathcal{A}} \mathbf{E}\left[\int_s^\tau f(Y_t,u_t)\,dt\right], \tag{74}$$

*with*

$$\begin{cases} dY_t = \mu(Y_t,\, u_t)\,dt + \sigma(Y_t,\, u_t)\,dB_t, \\ Y_0 = y\ . \end{cases}$$

*Suppose that* $V \in C^2(\mathbb{R}^+)$ *satisfies*

$$\mathbf{E}\left[|V(Y_\alpha)| + \int_0^\alpha |L^v V(Y_t)|\,dt\right] < +\infty,$$

*for all bounded stopping times* $\alpha < \tau$*, for all* $y \in \mathbb{R}$ *and all* $v \in \mathcal{A}$*, where*

$$(L^z V)(s,y) := \frac{\partial V(s,y)}{\partial s} + \mu(y,v)\frac{\partial V}{\partial y} + \frac{\sigma^2(y,v)}{2}\frac{\partial^2 V}{\partial y^2}.$$

*Moreover, suppose that an optimal control* $u^*$ *exists, then we have*

$$\sup_{v \in \mathcal{A}}\{f(y,v) + (L^v V)(y)\} = 0, \tag{75}$$

*and the supremum is obtained if it yialds* $v = u_t^* = u^*(t)$*, that is*

$$f(y,u^*(t)) + (L^{u^*(t)}V)(y) = 0\,.$$

The theorem 5 also applies to the corresponding *minimum* problem

$$\phi(s,y) := \inf_{u_t \in \mathcal{A}} \mathbf{E}\left[\int_s^\tau f(Y_t,u_t)\,dt\right].$$

We have in fact

$$\phi(s,y) = -\sup_{u_t \in \mathcal{A}} \mathbf{E}\left[\int_s^\tau -f(Y_t,u_t)\,dt\right],$$

from which, by replacing $V$ with $-\phi$ and $f$ with $-f$, it follows that the (75) in the theorem 2 becomes

$$\inf_{v \in \mathcal{A}}\{f(y,v) + (L^z\phi)(y)\} = 0. \tag{76}$$

For the details of the proof of these two theorems we remind the reader to Øksendal (2003).

Appendix C

**Definition 1.** Given a set $\Omega$, a $\sigma - algebra$ $\mathcal{F}$ on $\Omega$ is a *family* $\mathcal{F}$ of subsets of $\Omega$ which fullfills the following properties:

(i) the empty set $\emptyset$ belongs to $\mathcal{F}$;

(ii) if $F \in \mathcal{F}$, then the complement $\bar{F}$ of $F$ in $\Omega$ belongs to $\mathcal{F}$, too;

(iii) if $A_1, A_2, A_3, \dots \in \mathcal{F}$, then $A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$. □

**Definition 2.** The pair $(\Omega, \mathcal{F})$ is called a *measurable space*. □

**Definition 3.** A *probability measure* $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$ such that

(i) $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$;

(ii) if $A_1, A_2, A_3, \dots \in \mathcal{F}$ and $A_i \cap A_j = \emptyset, \ \forall i \neq j$, then $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

□

**Definition 4.** The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*. It is called a *complete probability space* if $\mathcal{F}$ contains all subsets $S$ of $\Omega$ with $\mathbb{P} - outer\ measure$ zero, where the $\mathbb{P} - outer\ measure$, denoted by $\mathbb{P}^*$, is defined as

$$\mathbb{P}^*(G) = \inf \Big\{ \mathbb{P}(F) : F \in \mathcal{F} \text{ and } G \subset F \Big\} .$$ □

**Definition 5.** For a given *family* $\mathcal{G}$ of subsets of $\Omega$, the $\sigma - algebra$ denoted by the symbol $\mathcal{F}_{\mathcal{G}}$ and defined as

$$\mathcal{F}_{\mathcal{G}} = \bigcap \Big\{ \mathcal{F} \, : \, \mathcal{F} \text{ is a } \sigma - algebra \text{ of } \Omega \text{ and } \mathcal{G} \subset \mathcal{F} \Big\}$$

is called the $\sigma - algebra$ *generated by* $\mathcal{G}$. □

**Definition 6.** If $\Omega$ is a topological space (e.g. $\Omega = \mathbb{R}^n$) equipped with the topology $\mathcal{G}$ of all open subsets of $\Omega$, then the $\sigma - algebra$ $\mathcal{B} = \mathcal{F}_{\mathcal{G}}$ is called the *Borel* $\sigma - algebra$ on $\Omega$ and the elements $B \in \mathcal{B}$ are called *Borel sets*. □

**Definition 7.** Given the *measurable space* $(\Omega, \mathcal{F})$, the (increasing) family $\{\mathcal{M}_t\}_{t \geq 0}$ of $\sigma - algebras$ of $\Omega$ such that

$$\mathcal{M}_{t_1} \subset \mathcal{M}_{t_2} \subset \mathcal{F} \, , \qquad \forall \ 0 \leq t_1 < t_2 \, ,$$

is called a *filtration* on $(\Omega, \mathcal{F})$. □

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