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2.2 Mixed strategy Nash equilibrium in typical games

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2.5 Nash equilibrium: existence with infinite pure strategies sets

We study in this Lecture Note the notion of Nash Equilibrium (NE). We start in Section 2.1.1 by introducing the formal notion of Nash equilibrium, remarking the difference between rationalizable strategies and equilibrium strategies and then establish the relation between equilibrium strategy profiles and the solution of the system of best response correspondences. A few examples of determination of mixed strategy Nash equilibrium in typical matrix games are then examined in Section 2.1.2 for the purpose of clarifying the properties of the equilibrium points. The final Section 2.1.3 is dedicated to the consideration of some aspects concerning the epistemic foundations of Nash equilibrium. We distinguish here between a canonical approach, directly in line with Nash’s definition of an equilibrium strategy profile, and a Bayesian approach which results from the consideration of the players’ beliefs about each others’ choices introduces the idea of an equilibrium in beliefs. We illustrate in Section 2.2 a few examples of determination of degenerate and non degenerate MSNE. Sections 2.3
and 2.4 represent two approaches to the problem of existence of equilibrium in a finite normal form game, both proposed by John Nash in his two papers of 1950 and 1951. In the first the problem is tackled from the point of view of the existence of a solution to the set of the best response correspondences using the analytical tool of Kakutani fixed point theorem. In the second, the proof uses, as in Nash’ PhD dissertation, Brouwer Fixed point theorem and is based on the construction of an ingenious mapping, whose fixed point is a NE. We examine first in Section 2.3 this latter proof of existence, that Nash judged more fundamental, whose interest lies also in the fact that it was later used to prove the existence of a competitive general equilibrium. The alternative proof of existence requires the prior verification (Section 2.4.1) that the best response correspondences possess the properties needed for the application of Kakutani theorem. Section 2.5 concludes with a few considerations on the proof of equilibrium in games with an infinite set of strategies and an application to Cournot’s model of quantity oligopoly.

2.1 Nash equilibrium

2.1.1 Definition

Assume that the game $\Gamma = \{I, \Delta(S_i), u_i(\sigma_i, \sigma_{-i})\}$ contains only the rationalizable mixed strategies that have survived the elimination of mixed strategies that are not best response to some strategies of the other players. The set of rationalizable strategies can be very large, as we can immediately verify looking back to Bernheim’s game (Lecture Note n.1) which, after the iterated elimination of non-rationalizable strategies, is represented by the 3x3 payoff matrix of Fig. 2.1.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0 , 7</td>
<td>2 , 5</td>
<td>$\overline{7}$ , 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_2$</td>
<td>5 , 2</td>
<td>$\overline{3}$ , $\overline{3}$</td>
<td>5 , 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_3$</td>
<td>$\overline{7}$ , 0</td>
<td>2 , 5</td>
<td>0 , $\overline{7}$</td>
<td></td>
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</tbody>
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Fig. 2.1 – Bernheim’s game: all strategies are rationalizable

With the graphical expedient already introduced, a bar above a payoff shows that the corresponding strategy is rationalizable for player 1, while a bar under the payoff indicates that the corresponding strategy is rationalizable for player 2. We have already commented in Lecture Note n.1 on the notion of rationalizability. To repeat: a strategy is rationalizable if
there exists, on the one hand, a set of player \(i\)'s beliefs about a possible mixed strategy of the other players that makes a given strategy, pure or mixed, a best response to those conjectures and, on the other hand, a corresponding set of other players’ beliefs about \(i\)'s possible mixed strategies that would make their choices a best response to those conjectures. We have remarked that, in order to rationalize strategies \(a_1\) and \(a_3\) of player 1 and of strategies \(b_1\) and \(b_3\) of player 2, it is necessary to resort to an endless cyclical chain of beliefs about beliefs, in which at each step one of the two players’ conjectures are systematically disproved. The resulting strategy profiles could hardly qualify as sensible equilibrium points of the game. The strategy profile \((a_2, b_2)\) stands up, on the contrary, for the property that the players’ conjectures are verified: while rationalizability allows players to have misconceptions about each others’ beliefs, Nash equilibrium excludes the possibility of any such misconception. This is, indeed, the characterizing feature of Nash equilibrium, graphically identified in the matrix of Fig. 2.1 by the cell carrying both an upper and a lower bar on the payoffs.\(^1\) Formally

**Definition 2.1.** A mixed strategy profile \((\sigma_i^*, \sigma_{-i}^*) \in \Delta(S)\) is a Nash equilibrium (NE)\(^2\) of the simultaneous move game \(\Gamma = \{I, \Delta(S_i), u_i(\sigma_i, \sigma_{-i})\}\) if

\[
\forall i \in I \text{ and } \forall \sigma_i \in \Delta(S_i) \quad u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i, \sigma'_{-i}) \quad \forall \sigma'_i \neq \sigma_i
\]

In words, in the equilibrium position:

- every player maximizes his expected payoff;
- chooses a strategy that is a best response;
- to the best responses of all the other players.

As a strategy profile representing each player’s best response to the best responses of all other players, NE can be viewed as the solution of the system of the players’ best response correspondences.

In *Lecture Note* n. 1 we have defined player \(i\)'s best response correspondence as

\[
BR_i(\sigma_{-i}) = \left\{ \sigma_i \in \Delta(S_i) \mid u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \quad \forall \sigma'_i \neq \sigma_i \right\} =
\]

\[
= \text{arg max}_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i})
\]

and accordingly determined a best response strategy as an element of the best response correspondence: \(\sigma_i \in BR_i(\sigma_{-i})\).

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\(^1\) The equilibrium position need not be unique, as shown by typical 2x2 games shortly to be considered.

\(^2\) With a hopefully excusable abuse of language, the initials NE will indicate both the instance of a single equilibrium position as well as the case of multiple equilibria. The context should avoid the risk of misinterpretation.
Definition (2.2) is perfectly general in the sense that it applies to any \( \sigma_{-i} \) and thus, in particular, for \( \sigma_{-i}^* \) and for all \( i \in I \). \(^3\) We can, therefore, define the Nash equilibrium strategy profile \( (\sigma_i^*, \sigma_{-i}^*) \) as the set of strategies that satisfy the relations

\[
\sigma_i^* \in BR_i(\sigma_{-i}^*) \quad \text{for all } i
\]

and establish the following proposition.

**Proposition 2.1** The strategy profile \( (\sigma_i^*, \sigma_{-i}^*) \) is a Nash equilibrium of the game \( \Gamma = (I, \Delta(S_i), u_i(\sigma_i, \sigma_{-i})) \) if and only if it is a solution of the set of correspondences (2.2).

**Proof** Let us consider first the “if” part (i.e. the necessary condition). From (2.2) and (2.3) \( \sigma_i^* \in BR_i(\sigma_{-i}^*) = \arg \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) \); hence \( u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \) for all \( i \). \( (\sigma_i^*, \sigma_{-i}^*) \) is, therefore, a Nash equilibrium. Let us consider now the “only if” part (i.e. the sufficient condition). Suppose that the strategy profile \( (\sigma_i^*, \sigma_{-i}^*) \) is a Nash equilibrium, then for all \( i \) \( \sigma_i^* \in \arg \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) = BR_i(\sigma_{-i}^*) \) and is therefore a solution of the system of correspondences (2.3).

Postponing a presentation of the proofs of existence of NE in finite simultaneous-move games, we will first use the result of Proposition 2.1, namely the property that NE is determined by the intersection points of the players’ best response correspondences, to study the variety of equilibrium solutions of some typical games. A minimum of familiarity with this issue is necessary in order to better follow the discussion on the epistemic foundations of NE.

### 2.1.2 Mixed strategy Nash equilibrium in typical games

We start with the determination of NE in the three games already examined: Bernheim’s, *Matching Pennies* and *Prisoners’ Dilemma*.

With regard to Bernheim’s example (see Fig. 1.13 in *Lecture Note n.1*), we have fixed the attention on the determination of the rationalizable pure strategies indicated in Fig. 2.1 above. Since an uncountable number of mixed strategies can obviously be obtained “mixing” the pure strategies of both players, the rationalizability approach to the solution of strategic games

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\(^3\) The properties of the best response correspondence will be discussed in Section 2.4.
initially focused on pure strategies solutions.\textsuperscript{4} The set $S_N$ of NE, in this case unique, is contained in the set of rationalizable pure strategies $S_R$: $S_N \subseteq S_R$.

Turning to the game of \textit{Matching Pennies} we can refer back to the determination of best response strategies defined by relations (1.19) and (1.21) and to Fig. 1.18 of \textit{Lecture Note n.1}. Both strategies \textit{Heads} and \textit{Tails} are rationalizable for each player: again this requires a cyclical chain of beliefs about the other player’s beliefs. The set of rationalizable mixed strategies coincides therefore with the Cartesian product $\Delta(S)$ of both players’ mixed strategy sets. NE is the unique mixed strategy profile $\left(\sigma_1^* = \sigma_2^* = \frac{1}{2}\right)$, graphically the intersection of the best correspondences of the two players. Any departure by either player from this unique mixed strategy equilibrium would lead both players to depart in a inconsistent way to one of their pure strategies.

In \textit{Lecture Note n.1} the best response strategies of the \textit{Prisoners’ Dilemma} game have been determined. Fig. 1.19 shows the NE in strict dominance as the degenerate mixed strategy equilibrium $\left(\sigma_1^* = \sigma_2^* = 0\right)$.

We consider now two coordination games. These are games in which, due to the presence of multiple NE, a problem arises of coordination of the players’ independent choices on the same profile of equilibrium strategies. Rousseau illustrates the benefits of cooperation which would ensue from coordination on the socially preferable (i.e. Pareto superior) strategy profile with the following pedagogical story. Two hunters must decide simultaneously whether to hunt for a stag or for a hare. The hunters can take the stag, an important prey, only if they hunt together, while everyone can take a hare, a modest prey, if they hunt separately. If one hunts for the hare and the other one for stag, the former gets his prey and the other remains empty handed.

The story easily lends itself to a presentation in game form, the \textit{Stag-Hunt} game. Consider the following version (version 1) of the (see Fig. 4.1.A). There are only two players, $i = 1, 2$, with the same pure strategy set: hunt the stag or hunt the hare, for short $S_i = \{\text{Stag}, \text{Hare}\}$. The payoffs reflect Rousseau’s story: in order to catch the stag the hunters must join efforts and together hunt for it; cooperation is rewarded with a payoff of 4 to each of them. Hunting separately, each can get a hare with a payoff of only 1. If one goes alone after the stag and the other hunts the hare, the former’s payoff is zero. Fig. 2.2 shows the resulting payoff matrix of the game.\textsuperscript{5}

\textsuperscript{4} We will show in a subsequent \textit{Lecture Note} that the set of rationalizable strategies is the conve hull of the Nash equilibria of the game

\textsuperscript{5} We will use in a further \textit{Lecture Note} different versions of the payoff matrix of the \textit{Stag-Hunt} game to illustrate the notions of payoff dominance and risk dominance.
<table>
<thead>
<tr>
<th></th>
<th>Stag (S)</th>
<th>Hare (H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stag (S)</td>
<td>4, 4</td>
<td>0, 1</td>
</tr>
<tr>
<td>Hare (H)</td>
<td>1, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

**Fig. 2.2 – The Stag-Hunt game**

Player 1’s best response correspondence is the solution of the following maximization problem

\[
\max_{\sigma_1 \in \Delta(S_1)} u_1(\sigma_1; \sigma_2) = 4\sigma_1\sigma_2 + (1 - \sigma_1) = (4\sigma_2 - 1)\sigma_1
\]

and therefore

\[
BR_1(\sigma_2) = \begin{cases} 
1 & \text{se } \sigma_2 \geq \frac{1}{4} \\ 
0 & \text{se } \sigma_2 < \frac{1}{4} 
\end{cases}
\]

Taking account of the symmetry of the game, fig. 2.3 depicts player 1’s and 2’s best response correspondences. The game has three Nash equilibria: two in pure strategies, \((\text{Stag, Stag})\) and \((\text{Hare, Hare})\), that is in the degenerate mixed strategies \((\sigma_1^* = \sigma_2^* = 1)\) and \((\sigma_1^* = \sigma_2^* = 0)\); and one in non degenerate mixed strategies \((\sigma_1^* = \sigma_2^* = \frac{1}{4})\).

As a second example of coordination game, we consider the game *Battle of the Sexes* of Fig. 2.4. Say that husband and wife have to choose whether to go the hokey game, which is the man’s preferred choice, or to the theater for a ballet show, which is the woman’s preferred choice, but both prefer to be together rather than being alone. The payoffs reflect the utilities
implied by these preferences; $\sigma_1$ and $\sigma_2$ are the probabilities that both players play the strategy Hockey.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<tbody>
<tr>
<td>Hockey</td>
<td>$\frac{2}{3}$, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Ballet</td>
<td>0, 0</td>
<td>$\frac{1}{2}$, $\frac{2}{3}$</td>
</tr>
</tbody>
</table>

**Fig. 2.4 – Payoff matrix of the game Battle of the Sexes**

The game has three Nash equilibria: the two pure strategy equilibria, $(H,H)$ and $(B,B)$, as well as a non degenerate mixed strategy equilibrium, that we can easily identify determining the best response correspondences of the two players. Following the same approach as with the *Stag-Hunt* game, the best response correspondence are for player 1

$$BR_1(\sigma_2) = \begin{cases} 1 & \text{se } \sigma_2 \geq \frac{1}{3} \\ 0 & \text{else} \end{cases}$$

and for player 2

$$BR_2(\sigma_1) = \begin{cases} 1 & \text{se } \sigma_1 \geq \frac{2}{3} \\ 0 & \text{else} \end{cases}$$

**Fig. 2.5 – Battle of the Sexes: Best response correspondences and mixed strategy NE**
Fig. 2.5 gives a graphical representation of these correspondences. The three intersection points correspond to the three equilibria; two are in degenerate mixed strategies – the equilibrium profile \((H,H)\), that is \((\sigma_1^* = 1; \sigma_2^* = 1)\), and the equilibrium profile \((B,B)\), that is \((\sigma_1^* = 0; \sigma_2^* = 0)\) - and one is non degenerate mixed strategy \(\left(\sigma_1^* = \frac{2}{3}; \sigma_2^* = \frac{1}{3}\right)\).

Also the game \textit{Battle of the Sexes} presents a coordination problem, but of a different nature. In the \textit{Stag-Hunt} game the pure strategy NE can be Pareto ranked, whereas in the \textit{Battle of the Sexes} game the pure strategy solutions cannot be Pareto ordered. One may think that in the first case the players ought to coordinate their independent choices on the payoff dominant solution, in the second case an easily conceivable signal may provide a proper alternating criterion.\(^6\)

2.1.2 Epistemic foundations

Having defined Nash equilibrium as a strategy profile that is a best response to the best response of all players, it is time to pose the question of what properties should the actions chosen by a rational player have to satisfy in order to avoid being self-defeating, that is in order to exclude the presence of an incentive to behave differently? There are two approaches.\(^7\)

The first, here called the \textit{canonical approach} because directly in line with Nash definition of the equilibrium points, is in terms of the properties of the actions/strategies/plans of action of the players. What knowledge and in force of what reasoning of each others’ actions can one justify the players’ choice of a Nash equilibrium strategy profile? In other words, what makes such a choice rationally self-enforcing?

The second, here called the \textit{Bayesian approach} because the result of the direct extension to situations of strategic interaction of the Bayesian view of rational behavior under uncertainty, is in terms of the beliefs/conjectures/assessments that each player has to make concerning the action choices of the other players. Then, again, what knowledge and in force of what reasoning of each others’ beliefs can one justify the players’ choice of a Bayes-Nash equilibrium strategy profile? In other words, what makes a beliefs equilibrium rationally self-enforcing?

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\(^6\) We will analyze both these criteria of equilibrium selection in the sequel of the course.

\(^7\) The distinction and, at the same time, the parallel nature of the two approaches is made very clear by Hillas and Kohlberg in their survey of the “Foundations of Strategic Equilibrium” (2002). We have taken the liberty to change the denomination of the approaches, to stress a first and a second phase of elaboration on the problem, but have preserved Hillas and Kohlberg’s qualification of a “self-enforcing” rational justification for the choice of an equilibrium strategy profile.
We turn to examine these two approaches in sequence with special attention for the first one.

2.1.3.1 The canonical approach

Nash’ definition of equilibrium points identifies the task of game theory with the determination of the strategies/actions that individually rational players would or should – according to the positive or to the normative view of noncooperative game theory – adopt knowing the structure of the game and recognizing the rationality and the knowledge of the other players. The specification of the equilibrium strategies is thus defined as a self-enforcing strategy profile that players plan to use in the play of the game; in other words, the equilibrium strategies are interpreted as ipso facto providing an implicit analysis of the game-theoretic rationality foundations of the players’ behavior. Luce and Raiffa, in their amply referenced to Games and Decisions (1957), and James Friedman, in the context of oligopoly theory, openly validate this line of interpretation.

Pointing to the key role, in the determination of the equilibrium points, of the absence of any incentive to deviate from the recommendation of the theory, Luce and Raiffa write: “if our non-cooperative theory is to lead to an n-tuple of strategy choices and if it is to have the property that knowledge of the theory does not lead one to make a choice different from that dictated by the theory, then the strategies isolated by the theory must be equilibrium point” (p. 173). Friedman, on the other hand, commenting on Cournot equilibrium, a clear forerunner of Nash solution, writes: “The beauty of [Cournot equilibrium] lies in its inherent believability. Nowhere is there false information or firms acting on the premise that their rivals are less perceptive than they. ... Put another way, if you impose some standards of reasonableness on your own choice and assume others will use these standards too, then an output vector which is not a Cournot equilibrium will not look reasonable” (1977, pp. 142-143).

---

8 The idea of a self-enforcing plan of actions is Hillas and Kohlberg’s (2002) idea of Nash equilibrium. The notion of equilibrium as a self-enforcing position is, however, not uncontroversial (see in particular Kreps (1990) and Aumann (1990)).
9 See Stalnaker (1994)
10 The idea that players should have no incentive to depart from the recommendations of the theory was first formulated by von Neumann and Morgenstern who suggest that game theorists, in an outside position and knowing the players’ situation, can tell the players what the theory recommends.
11 In this context “reasonable” is clearly synonymous of “rational”.
12 We ask the readers’ permission for a comment stimulated by the reading of Johansen’ paper: radically different views about the role of the rationality assumption in economic theory lead to equally radically disparate judgments of the work of past economists. Opposite to Friedman’s perceptive appreciation of Cournot’s equilibrium in his oligopoly model, stands Simon’s embarrassingly ungenerous critique of Cournot, accused to have “identified a problem that has become the permanent and ineradicable scandal of economic theory” (Simon 1977, p. ). Frankly, the defense of procedural rationality ought to find better arguments!
The status of Nash equilibrium from the standpoint of game theoretic rationality in noncooperative game theory is first fully analyzed by Leif Johansen (1982), who finds explicit support for his argument in the just quoted positions expressed by Luce and Raiffa and by Friedman. Johansen formulates a set of postulates, that he deems necessary and sufficient to conclude that, if the Nash equilibrium is unique, then “the Nash behavior is the individually rational behavior” (p. 436, italics in the original).

Johansen proposes the following postulates as a supporting epistemic basis for this view of Nash equilibrium as a self-enforcing plans of actions.:  

**Postulate 1.** A player makes his decision ... on the basis of, and only on the basis of information concerning the action possibility sets ... and the preference functions of all players.

Postulate 1 implies that a player knows the structure of the game. Mixed strategies, considered here to be the result of a deliberate randomization chosen by the player, are explicitly included in the strategy set.

**Postulate 2.** In choosing his own decision, a player assumes that the other players are rational in the same way as he himself is rational.

In the Author’s words: “This postulate implies a sort of symmetry. All players are rational in the same way, and it is then part of the rational behavior of the individual player to recognize and take into account the rationality of other players”.

A more common formulation of the rationality implications of Postulates 1 and 2 would be the assumption of individual rationality and of common knowledge of rationality of the players and of the structure of the game.

**Postulate 3.** If some decision is the rational decision to make for an individual player, then this decision can be correctly predicted by other players.

Postulate 3 draws the implications of Postulates 1 and 2. First, if a player, analyzing the game, finds that “a certain action is the rational decision on his part, then other players can imagine themselves in his place and duplicate his analysis”. Second, and symmetrically, since every “player can predict the decisions that will be taken by other players, … he also knows … that the other players can predict his own decision”.

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13 Actually Harsanyi (1966) had already studied the problem of rational behavior in general game situations, considering cooperative, noncooperative and bargaining games. Such a very general point of view does not appear to be appropriate to the specific issues posed by noncooperative games.

14 Postulates and added quoted clarifying explanations are from pages 432-434 of Johansen’s paper.
Postulate 4. Being able to predict the actions to be taken by other players, a player's own decision maximizes his preference function corresponding to the predicted actions of other players.

As a consequence of Postulates 1-3, Postulate 4 corresponds to the "rationality postulate of noncooperative behavior" mentioned above.

Postulates 3 and 4 have been differently formulated as: the Nash equilibrium assumption, the consistent alignment of beliefs, the complete knowledge of each player’s of the other players’ strategy choices, the assumption of perfect foresight. The last formulation, taken from Mas Colell’s perceptive remarks on Nash equilibrium and economics, very directly and effectively renders the condition of absence of misconceptions.

Johansen’s contribution has not gone unnoticed; it has actually been the object of an intense subsequent debate, the motivations of which can be traced principally to the following points: non uniqueness of Nash equilibrium; Bernheim and Pearce’s influential criticism of Nash equilibrium and their forceful stand in favor of rationalizability as the solution concept consistent with the assumption of common knowledge of rationality; and the Bayesian reinterpretation of mixed strategies as an expression of every player’s uncertainty about the strategy choices of the other players.

Johansen himself has underscored the problems that arise when Nash equilibrium fails to be unique. A coordination problem arises: the four Postulates do not provide any criterion for a rational, mutually consistent strategy choice among multiple equilibrium points. On the issue of equilibrium selection in coordination games Johansen’s conclusion is negative: “A convincing general theory of what an individually rational action is in cases where we encounter the multiplicity problem, is hard to imagine”. This conclusion may actually be stronger than warranted in view of the consideration of the several accepted refinements of Nash equilibrium, which will be examined in the sequel of these Notes. Johansen makes, however, a second and more pressing observation: he feels that the multiplicity of equilibria risks of jeopardizing the basic principle of individual rationality in economics when “we try to extend it into … [the] fields of economic interaction” (p. 437). This calls for extending to the context of strategic uncertainty the Bayesian analysis of individual rational decision-making under uncertainty. We will be briefly go back to this question on the point concerning the interpretation of mixed strategies.

Bernheim’s (1986) criticism of Nash equilibrium and of Johansen’s defense focuses essentially on the question of multiplicity. If the stated predictability (Postulates 3 and 4 above) of every player’s logical construction leading to the choice of a given strategy is

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16 Bernheim (1986) has later proved the consistency of rationalizability with Aumann’s notion of correlated equilibrium, an extension of Nash equilibrium.
conditioned by the existence of a unique equilibrium point, then Johansen’s proof – and this is Bernheim’s point - ought to contain a postulate excluding multiplicity as part of the rationality of players’ choices. But this is not so in Johansen’s model of rationality. As Bernheim aptly remarks, one could, however, consider the set of Nash equilibria restricted to the strategy profiles consistent with the various refinements proposed by the literature. The active research of refinement aims at the specification of further postulates to be added to the above list, in order to eliminate multiplicity and to identify a unique solution. Not all refinements are, however, on an equal footing: some may indeed be taken as expressions of a strictly rational behavior and easily accommodated in an enlarged set of postulates; others appear to be more the result of plausible considerations than of the identification of features of strictly rational behavior.

Bernheim’s conclusion is then that refinements do not provide a convincing rational foundation to Nash equilibrium. The systematic introduction of elements of uncertainty in the definition of the structure of the games, with the associated adoption of the extended notion of Bayesian Nash equilibrium, seems to offer a better perspective and new insights capable of impacting on the problem of rationality in situations of strategic interaction.

The critique of Nash equilibrium because of the multiplicity of solutions does not, however, justify the claim that we should forgo Nash’s solution, as an epistemic foundation of noncooperative game theory, in favor of the approach that maintains that the assumption of common knowledge of rationality and of the structure of the game does not permit to go beyond rationalizability as a solution criterion in strategic situations. Two observations are here in order. First, as Bernheim’s example shows, in many games rationalizability implies few, if any, restrictions; second, and more importantly, contrary to rationalizable strategy profiles, Nash equilibria are the result of situations of absence of misconceptions of the players’ reciprocal beliefs. The properties of Nash equilibria are, therefore, substantially different from those of rationalizable strategies.

2.1.3.1 The Bayesian approach

The final point regards the interpretation of mixed strategies and the consequent impact on the epistemic foundations of Nash equilibrium. In Johansen’s work mixed strategies are viewed as deliberate choices of randomization made by each player in order to dodge the risk of being exploited by the opponents. They are as such included in every player’s strategy set with no distinction from pure strategies.

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17 It has been shown that the set of rationalizable strategy profiles coincides with the set of correlated equilibria of a noncooperative game. We defer a brief examination of this issue to the Lecture Note dedicated to the study of the notion of correlated equilibrium.
The study of noncooperative game theory from a decision-theoretic standpoint has changed the approach. The Bayesian view of individual decision-making under exogenous uncertainty has been extended to the study of individual decision-making under strategic uncertainty, namely the uncertainty concerning the other players’ strategy choices. Mixed strategies are thus considered to be the expression of the subjective (probabilistic) beliefs that players $j \neq i$ have about $i$’s strategy choice. A new element enters into the description of the rationality of the players and their epistemic state: “what they know or believe about the game and about each other’s rationality, actions, knowledge, and beliefs” (Aumann and Brandenburger, 1995, p. 1161).\(^{18}\)

Johansen’s Postulates 3 and 4, which imply that common knowledge of the equilibrium strategies of the players is part of the assumption of common knowledge of rationality, must therefore be changed into the requirement of common knowledge of beliefs concerning the actions of the other players. Assume that the only strategic uncertainty concerns the set $S_i \times S_{-i}$ of the possible strategy profiles of the game\(^ {19}\) and that the players’ beliefs can be expressed by a subjective probability distribution on this set. We then have, borrowing Stalnaker’s (1994, pp. 60-61) compact formulation, the following Postulate that replaces Johansen’s Postulates 3 and 4.

**Postulate 3'4'**. Players have complete knowledge of each player’s beliefs about the other’s strategy choices.

The relation between the Bayesian approach and the Nash definition of mixed strategy equilibrium points is spelled out in detail by Hillas and Kohlberg (2002, p. 1655) in the following

**Proposition.** Suppose that each player knows the game being played, knows that all the players are rational, and knows the conjectures of the other players. Suppose also that

(a) The conjecture of any two players about the choices of a third one are the same, and
(b) The conjectures of any player about the choices of two other players are independent.

Then the vector of conjectures about each player’s choice forms a strategic equilibrium.

The key to the proof lies in the recognition that, on account of suppositions (a) and (b), the conjectures are a vector of $\sigma$ of mixed strategies in Nash sense. Then the rationality assumptions in the premises imply that player $i$’s choices should put a positive weight –

\(^{18}\) The analytical problems raised by the infinite regress of beliefs of the players have been studies and solved by Zamir and Mertens (1985), who have shown the existence of a general type space. This leads to the possibility of defining a Bayesian decision problem associated to the game. See also Tan and Werlang (1988).

\(^{19}\) Thus excluding the further uncertainty due to incomplete information about the structure of the game.
should assign a positive $\sigma_i$ - only on those strategies that maximize his expected payoff given his conjecture $\sigma_{-i}$ of the others’ choices. Thus $(\sigma_i, \sigma_{-i})$ is a strategic equilibrium.

### 2.2 Properties of a non degenerate Nash equilibrium in mixed strategies

A non degenerate Nash equilibrium in mixed strategies has an important property, which justifies the recourse to a short cut to determine the mixed strategy equilibrium. A bit informally, this property says that, if the pure strategies $s_i$ and $s'_i$ are assigned positive probability in the mixed strategy equilibrium $(\sigma_i^*, \sigma_{-i}^*)$, then their expected payoff must be equal: $u_i(s_i, \sigma_{-i}^*) = u_i(s'_i, \sigma_{-i}^*)$ and, obviously, equal to the expected payoff of the mixed strategy profile $u_i(\sigma_i^*, \sigma_{-i}^*)$.

To verify the validity of this statement, take again the game *Battle of the Sexes* and consider the non degenerate mixed strategy equilibrium $(\sigma_1^*= \frac{2}{3}; \sigma_2^*= \frac{1}{3})$. By direct calculation we have

$$(2.8) \quad u_i(\sigma_i^*; \sigma_2^*) = u_i(Ballet; \sigma_2^*) = \frac{2}{3}$$

**Proposition 2.2** Let $S_i^+ \subset S_i$ be the set of pure strategies that player $i$ plays with positive probability in the mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$. Strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a Nash equilibrium of game $\Gamma = \{I, \Delta(S_i), u_i(\sigma_i^*, \sigma_{-i}^*)\}_{i=1}^I$ if and only if for all $i=1, \ldots, I$

$$(2.9) \quad u_i(s_i, \sigma_{-i}^*) = u_i(s'_i, \sigma_{-i}^*) \quad \text{for all } s_i, s'_i \in S_i^+;$$

and

----

20. See MWG. (1995, p. 250, Proposition 8.D.1). Proof of the proposition is by contradiction. For the *if* part, suppose that $(\sigma_i^*, \sigma_{-i}^*)$ is a Nash equilibrium in non degenerate mixed strategies, but that (2.10) is not satisfied. Then either $u_i(s_i, \sigma_{-i}^*) > u_i(s'_i, \sigma_{-i}^*)$ or $u_i(s_i, \sigma_{-i}^*) < u_i(s'_i, \sigma_{-i}^*)$. In either case there is a pure strategy offering a higher payoff than the mixed strategy $(\sigma_i^*, \sigma_{-i}^*)$, which cannot be a Nash equilibrium. For the *only if* part, assume that (2.10) and (2.11) are satisfied, but the resulting mixed strategy is not a Nash equilibrium. There must then be some player $i$ who has a, degenerate or non degenerate, mixed strategy $\sigma_i'$ such that $u_i(\sigma_i', \sigma_{-i}^*) > u_i(\sigma_i^*, \sigma_{-i}^*)$. But in this case there must exists a pure strategy $s_i'$, which is played with positive probability in the mixed strategy $\sigma_i'$ such that $u_i(s_i', \sigma_{-i}^*) > u_i(\sigma_i^*, \sigma_{-i}^*)$, contradicting the assumption that (2.10) is satisfied.

21. A more correct wording, in line with the Bayesian approach, would be: “Let $S_i^+ \subset S_i$ be the set of pure strategies that players $j \neq i$ believe that player $i$ would assign a positive probability in the mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$. 

---
As a result of proposition 2.2 there is an easy way to determine, if it exists, a non degenerate mixed strategy solution. If pure strategies \( s_i \) and \( s'_i \) are both played with positive probability in the Nash equilibrium \((\sigma^*_i, \sigma_{-i}^*)\), then \( \sigma_{-i}^* \) must satisfy the condition \( u_i(s_i, \sigma_{-i}^*) = u_i(s'_i, \sigma_{-i}^*) \) and vice versa.

Consider, by way of example, the non degenerate mixed strategy solution of the game *Battle of the Sexes*. Player 1 must be indifference between Hockey and Ballet. This obtains if the expected payoffs, which depend on the mixed strategy of player 2, are equal

\[
(2.11) \quad 2\sigma_2 = 1 - \sigma_2
\]

Similarly for player 2; indifference obtains if the expected payoffs of his pure strategies, which now depend on player 1’s mixed strategy, are equal

\[
(2.12) \quad \sigma_1 = 2\left(1 - \sigma_1\right)
\]

According to Proposition 2.2, the values of \( \sigma_1 \) and \( \sigma_2 \) that satisfy (2.11) - \( \sigma_2 = \frac{1}{3} \) - and (2.12) - \( \sigma_1 = \frac{2}{3} \) - constitute the non degenerate mixed strategy Nash equilibrium, that we have already determined.

As highlighted by Proposition 2.2, in a non degenerate mixed strategy equilibrium each player must be indifferent between the pure strategies that are assigned a strictly positive probability by the equilibrium mixed strategy. This analytical result prompts a specific criticism of mixed strategy equilibrium: if, say, player \( i \), is indifferent between his pure strategies \( s_i \) and \( s'_i \), why should he play his mixed strategy rather than one of the pure strategies? In fact, this criticism is not so much against the play of mixed strategies as it is more directly to the notion itself of Nash equilibrium. It is obvious that if player \( i \) does not play his part of the mixed strategy equilibrium and deviates to one of the pure strategies \( s_i \) or \( s'_i \), there is no reason to expect that player 2 will play his part of the mixed strategy equilibrium. Fig. 2.5 is, in this respect, most clear: if player 1 plays the pure strategy *Hockey*, player 2’s best response would be to play his pure strategy *Hockey*, and not the equilibrium mixed strategy. The criticism is thus really to the key assumption of equilibrium, which is the distinguishing feature of Nash equilibrium.

### 2.3 Existence of Nash equilibrium in games with finite strategy set: the Brouwer approach
Nash has proposed two proofs of the existence of equilibrium for the game 
\[ \Gamma = \{ I, \Delta(S_i), u_i(\sigma, \sigma_{-i}) \} \] with the key assumptions that the number of players as well as
the number of pure strategies is finite. In the short paper published in 1950, he indicates
that the conditions are met for the application of Kakutani’s fixed point theorem. In the
1951 paper, he shows the existence of an equilibrium solution using Brouwer’s fixed point
theorem, which he considers an improvement with respect to the preceding proof. Both
techniques have been extensively used to show the existence of a general competitive
equilibrium.

We start by presenting Nash’s proof of existence of equilibrium based on the use of
Brouwer fixed point theorem.

**Brouwer’s fixed point theorem.** Let \( A \subset \square^N \) be a non empty, compact and convex
set; let \( f : A \rightarrow A \) be a continuous function of \( A \) into itself. Then \( f(\cdot) \) has a fixed
point: i.e., there is an \( x \in A \) such that \( f(x) = x \).

The proof is immediate for \( N = 1 \) and \( A = [0,1] \); it is then a simple consequence of the
intermediate value theorem.\(^{22}\) The proof is, however, quite complex for \( N \geq 2 \). We will not
go into it and move directly to Nash’s use of Brouwer’s theorem.

The proof of the existence of Nash equilibrium is articulated in the following three steps:

- Step 1. Construct a mapping \( T(\sigma_i, \sigma_j) \) from the set of mixed strategies \( \Delta(S) \) into itself.
  
  This will be done considering all possible deviations to one or more pure strategies starting
  from any possible mixed strategy profile.

- Steps 2. Show that this mapping satisfies the conditions for the application of Brouwer’s
  theorem and thus admits a fixed point.

- Step 3. Prove Nash Theorem, namely that the fixed point of the mapping is a NE and that
  the Nash equilibrium must be a fixed point of the mapping.

**Step 1.**

Simplifying Nash’s approach, let us assume that there only two players, \( i \) and \( j \).\(^{23}\) Let

\[
c_i(s_i^k, \sigma_j) = \max \left\{ 0, u_i(s_i^k, \sigma_j) - u_i(\sigma_i, \sigma_j) \right\}
\]

(2.13)

\(^{22}\) See MasColell et al. (1995, pp. 952-3).

\(^{23}\) We follow the presentation by Luce and Raiffa (1957, pp.391-3 where a hint is also offered as to the plausibility
of Brouwer’s theorem for \( N = 2 \)).
be the additional payoff obtaining for player $i$ from a deviation from the mixed strategy $\sigma_i$ to the pure strategy $s_i^k$. Define similarly

$$d_j(\sigma_i,s_j^k) = \max \left\{ 0, u_j(\sigma_i,s_j^k) - u_j(\sigma_i,\sigma_j) \right\}$$

as the gain of player $j$ of a deviation from the mixed strategy $\sigma_j$ to the pure strategy $s_j^k$.

Define now the mapping $T(\cdot): (\sigma_i,\sigma_j) \rightarrow (\sigma_i',\sigma_j')$ from the mixed strategy $(\sigma_i,\sigma_j)$ to the mixed strategy $(\sigma_i',\sigma_j')$ as

$$\sigma_i' = \frac{\sigma_i + c_i(s_i^k,\sigma_j)}{1 + \sum_{s_i \in S_i} c_i(s_i^k,\sigma_j)}$$

$$\sigma_j' = \frac{\sigma_j + d_j(\sigma_i,s_j^k)}{1 + \sum_{s_j \in S_j} d_j(\sigma_i,s_j^k)}$$

(2.15)

*Step 2*

Let us first verify that $(\sigma_i',\sigma_j') \in \Delta(S)$. Note, to this end, that the mapping $T(\cdot)$ is

- non empty, since $(\sigma_i,\sigma_j) \in \Delta(S)$;
- non negative, since both the numerator and the denominator are non negative;

- compact, since $\sum_{s_i} \sigma_i' = \sum \sigma_i + \sum c_i = 1$ because $\sum \sigma_i = 1$.

Furthermore, the mapping $T(\cdot)$ is continuous in $(\sigma_i,\sigma_j)$ since both $c_i$ and $d_j$ are continuous in the other player’s mixed strategy.

The mapping $T(\cdot)$ is therefore a continuous mapping of the non empty, compact and convex set of mixed strategies $\Delta(S)$ into itself. The mapping $T(\cdot)$ meets therefore the conditions for the application of Brouwer’s fixed point theorem: there exists, therefore, a $(\sigma_i,\sigma_j) \in \Delta(S)$ such that all deviations $c_i$ and $d_j$ are equal to zero.

*Step 3*

**Proposition 2.2 (Nash Theorem)** The strategy profile $(\sigma_i,\sigma_{-i})$ is a Nash equilibrium of the game $\Gamma = \{I,\Delta(S_i),u_i(\sigma_i,\sigma_{-i})\}$ if and only if it is a fixed point of the mapping (2.15).
Proof. For the “if” part, assume that \((\sigma_i, \sigma_j)\) is a Nash equilibrium, that is \((\sigma_i, \sigma_j)\) is an optimal strategy for both players. This means that there cannot be any pair of pure strategies \((s_i, s_j)\) giving a higher payoff. It follows that for all possible deviations to pure strategies the corresponding benefits \(c_i(s^*_i, \sigma_j)\) and \(d_j(\sigma_i, s^*_j)\), as defined in (2.13) and (2.14), are all zero. Hence \((\sigma_i, \sigma_j)\) is a fixed point of \(T(\cdot)\).

For the “only if” part, assume that \((\sigma_i, \sigma_j)\) is a fixed point of \(T(\cdot)\). The payoff of player \(i\) of his pure strategy \(s^*_i\) is \(u_i(s^*_i, \sigma_j)\) whereas the payoff of his mixed strategy \(\sigma_i\) is \(u_i(\sigma_i, \sigma_j) = \sum s^*_i \sigma_i(s^*_i, \sigma_j) u_i(s^*_i, \sigma_j)\). Since \(u_i(\sigma_i, \sigma_j)\) is a weighted average of the payoffs of the pure strategies, there must be a pure strategy \(s^*_i\) used in the mixed strategy \(\sigma_i\) with a positive probability \(\sigma_i(s^*_i) > 0\) such that

\[
(2.16) \quad u_i(s^*_i, \sigma_j) \leq u_i(\sigma_i, \sigma_j)
\]

If \(s^*_i\) is the only pure strategy used in the degenerate mixed strategy \(\sigma_i\), then we can only substitute \(s^*_i\) for itself and (2.16) is verified with the equal sign. If, on the contrary, the two pure strategies \(s^*_i\) and \(s^*_h\) are both used with positive probability in the mixed strategy \(\sigma_i\), then \(u_i(\sigma_i, \sigma_j) = \sigma_i(s^*_i) u_i(s^*_i, \sigma_j) + \sigma_i(s^*_h) u_i(s^*_h, \sigma_j)\).

Therefore, either \(u_i(s^*_i, \sigma_j) = u_i(s^*_i, \sigma_j)\) and (2.16) is satisfied; or if, for instance, \(u_i(s^*_h, \sigma_j) > u_i(\sigma_i, \sigma_j)\), then necessarily \(u_i(s^*_i, \sigma_j) < u_i(\sigma_i, \sigma_j)\). This means that there is certainly a pure strategy \(s^*_i\) such that a deviation to that strategy meets condition (2.16). A similar reasoning can be made with regard to strategy \(s^*_h\) of player \(j\).

Let us then consider a deviation to strategies \(s^*_i\) and \(s^*_j\) for which n account of (2.16), and from definitions (2.13) and (2.14), we have

\[
c_i(s^*_i, \sigma_j) = \max \left\{ 0, u_i(s^*_i, \sigma_j) - u_i(\sigma_i, \sigma_j) \right\} = 0
\]

and

\[
d_j(\sigma_i, s^*_j) = \max \left\{ 0, u_j(\sigma_i, s^*_j) - u_i(\sigma_i, \sigma_j) \right\}
\]

Relations (2.15) become
Let now \((\sigma_i, \sigma_j)\) be a fixed point of \(T(\cdot, \cdot)\). Since it cannot be \((\sigma_i', \sigma_j') < (\sigma_i, \sigma_j)\), we must have \(\sum_{s_i} c_i \left( s_i^k, \sigma_j \right) = 0\) as well as \(\sum_{s_j} c_j \left( \sigma_i, s_j^k \right) = 0\). This shows that there is no profitable deviation from a fixed point of \(T(\cdot, \cdot)\), hence the fixed point is a Nash equilibrium.

2.4 The Kakutani approach to the existence of a Nash equilibrium

We now turn to a proof of Nash equilibrium using the analytical tool of Kakutani’s fixed point theorem, which generalizes Brouwer’s theorem to point-to-set correspondences. We begin by stating Kakutani’s theorem:

**Kakutani’s fixed point theorem.** Let \(A \subset \mathbb{R}^N\) be a non-empty, compact and convex subset of a finite dimensional vector space. Let \(\phi: A \to A\) be any upper-hemicontinuous point-to-set correspondence from \(A\) into itself with the property that for all \(x \in A\) the set \(\phi(x) \in A\) is non-empty, compact and convex. Then \(\phi(\cdot)\) has a fixed point: i.e., there is an \(x \in A\) such that \(x \in f(x)\).

According to Proposition 2.1, \((\sigma_i^*, \sigma_{-i}^*)\) is a Nash equilibrium strategy profile if it is a solution of the set of relations \(\sigma_i \in BR(\sigma_{-i})\) for all \(i\). In compact form

\[
\sigma \in BR(\sigma) = [BR_i(\sigma_{-i})]_{i=1}^I
\]

is then a system of point-to-set correspondences from \(\Delta(S)\) to \(\Delta(S)\). In order to make Kakutani’s theorem applicable for the proof of the existence of a solution, we must verify that all the conditions of the theorem are met. This is easily done given the properties of individual best response functions stated in section 1.3. By definition \(\Delta(S) = \times_{i} \Delta(S_i)\) is the Cartesian

2.4.1 Properties of best response correspondences of player \(i\)

Let us first consider the properties of best response correspondences of player \(i\) rewriting definition (2.2) using the following convenient matrix notation

\[
\sigma_i' = \frac{\sigma_i}{1 + \sum_{s_i \in S_i} c_i \left( s_i^k, \sigma_j \right)}
\]

\[
\sigma_j' = \frac{\sigma_j}{1 + \sum_{s_j \in S_j} d_j \left( \sigma_i, s_j^k \right)}
\]
(2.19) \[ BR_i^*(\sigma_{-i}) = \arg \max_{\Delta(S_i)} u_i(\sigma_i, \sigma_{-i}) = \arg \max_{\Delta(S_i)} \sigma_i \cdot U_i \sigma_{-i} \]

where \( U_i \) stands for the payoff matrix of any player \( i \).

As the maximum of the linear function \( u_i(\sigma_i, \sigma_{-i}) \) - continuous in the arguments \( \sigma_i \) and \( \sigma_{-i} \) and concave in \( \sigma_i \) - over the compact and convex set \( \Delta(S_i) \), the best response correspondence \( BR_i^*(\sigma_{-i}) \) has the following properties:

- non empty, by Weierstrass theorem;
- convex;
- upper-hemicontinuous.\(^{24}\)

To prove convexity of \( BR_i^*(\sigma_{-i}) \), let \( \sigma_i^{'}, \sigma_i^{''} \in BR_i^*(\sigma_{-i}) \). From the definition of best response we have

\[
(2.20) \quad u_i\left(\sigma_i^{'}, \sigma_{-i}\right) \geq u_i\left(\tau_i, \sigma_{-i}\right) \quad \text{for all } \tau_i \in \Delta(S_i)
\]

and for all \( \lambda \in [0,1] \)

\[
(2.21) \quad \lambda u_i\left(\sigma_i^{'}, \sigma_{-i}\right) + (1-\lambda) u_i\left(\sigma_i^{''}, \sigma_{-i}\right) \geq u_i\left(\tau_i, \sigma_{-i}\right) \quad \text{for all } \tau_i \in \Delta(S_i)
\]

By the linearity of \( u_i(\cdot) \)\(^{25}\)

\[
(2.22) \quad u_i\left(\lambda \sigma_i^{'}, (1-\lambda) \sigma_i^{''}, \sigma_{-i}\right) \geq u_i\left(\tau_i, \sigma_{-i}\right) \quad \text{for all } \tau_i \in \Delta(S_i)
\]

Hence \( \left(\lambda \sigma_i^{'}, (1-\lambda) \sigma_i^{''}, \sigma_{-i}\right) \in BR_i^*(\sigma_{-i}) \), thus showing that \( BR_i^*(\sigma_{-i}) \) is convex-valued.

To show that \( BR_i^*(\sigma_{-i}) \) is upper-hemicontinuous, suppose that \( \sigma_i^n \in \Delta(S_i) \) and \( \sigma_{-i}^n \in \Delta(S_{-i}) \) are convergent sequences with \( \lim \sigma_i^n = \sigma_i \) and \( \lim \sigma_{-i}^n = \sigma_{-i} \) with \( \sigma_i^n \in BR_i^*(\sigma_{-i}^n) \), that is with

\[
\lambda u_i\left(\sigma_i^n, \sigma_{-i}^n\right) + (1-\lambda) u_i\left(\sigma_i^{''n}, \sigma_{-i}^n\right) = \lambda \left[\sigma_i^n \cdot U_i \sigma_{-i}^n\right] + (1-\lambda) \left[\sigma_i^{''n} \cdot U_i \sigma_{-i}^n\right] \Rightarrow
\]

24 The correspondence \( \sigma_i \in BR_i^*(\sigma_{-i}) \) is upper-hemicontinuous (u.h.c.) if, given the sequence \( \sigma_{-i}^n \rightarrow \sigma_{-i} \), \( \sigma_i^n \in \Delta(S_i) \) and the sequence \( \sigma_i^n \rightarrow \sigma_i \), \( \sigma_i^n \in BR_i^*(\sigma_{-i}^n) \) with \( \sigma_i^n \in BR_i^*(\sigma_{-i}^n) \), then \( \sigma_i \in BR_i^*(\sigma_{-i}) \).

25 The result is immediate in matrix notation:

\[
\left[\lambda \sigma_i^{'}, (1-\lambda) \sigma_i^{''}\right] \cdot U_i \sigma_{-i} = \left[\lambda \sigma_i^{'}, (1-\lambda) \sigma_i^{''}\right] U_i \sigma_{-i} \Rightarrow
\]
\[(2.23)\quad u_i{(\sigma_i^n, \sigma_{-i}^n)} \geq u_i{(\tau_i, \sigma_{-i}^n)} \text{ for all } n \in \mathbb{N} \text{ and } \tau_i \in \Delta(S_i)\]

By the continuity of \( u_i \) on \( \Delta(S_{-i}) \) this in turn implies, for all \( \tau_i \in \Delta(S_i) \),

\[(2.24)\quad u_i{(\sigma_i, \sigma_{-i})} \geq u_i{(\tau_i, \sigma_{-i})} \text{ for all } \tau_i \in \Delta(S_i)\]

and shows that the correspondence \( BR_i(\sigma_{-i}) \) is upper-hemicontinuous.

Define now the set of best response correspondences (2.3) in the compact form

\[(2.25)\quad \sigma \in BR(\sigma) = \left[ BR_i(\sigma_{-i}) \right]_{i=1}^I\]

The set \( BR(\sigma) \) inherits the properties of each of the individual best response correspondences; \( BR(\sigma) \) is therefore:

- non empty;
- convex;
- upper-hemicontinuous

Hence \( BR(\sigma) \) is a mapping from \( \Delta(S) \rightarrow \Delta(S) \) with properties required for the application of Kakutani’s theorem. We conclude that the set of correspondences (2.25) has a fixed point and, as already proved in Proposition 2.1, that the strategy profile \( (\sigma_i^*, \sigma_{-i}^*) \) is a Nash equilibrium of the game \( \Gamma = \{ I, \Delta(S_i), u_i(\sigma_i, \sigma_{-i}) \} \) if and only if it is a solution of the set of correspondences (2.25).

### 2.5 Nash equilibrium: existence with infinite pure strategies sets

We investigate now the question of the existence of a Nash equilibrium in games in which the players have infinitely many pure strategies. This assumption is of particular interest in economics, since it is a standard assumption that quantities and prices be perfectly divisible and thus capable of assuming any value in a closed interval. Let accordingly the strategy set be of the form \( S_i = [\underline{s}_i, \overline{s}_i] \): a non empty, compact and convex set. Let, furthermore, the payoff functions \( u_i(s_i, s_{-i}) \) be continuous on the set \( S_\times = \times_i S_i \). We show, by an extension of Kakutani’s theorem, the existence of a pure strategy equilibrium if \( u_i(s_i, s_{-i}) \) is quasi concave.
Theorem (Debreu, Gklicsberg, Fan)\textsuperscript{26} Consider the strategic form game
\[ G = \{ I, S_i, u_i(s_i, s_{-i}) \} \] with a finite number of players. Assume that for every \( i \):

1. \( S_i \) is a non empty, compact and convex subset of a finite dimensional Euclidean space;
2. \( u_i(s_i, s_{-i}) \) is continuous in \( s_i \) and \( s_{-i} \) and quasi concave in \( s_i \).

Then the game \( G = \{ I, S_i, u_i(s_i, s_{-i}) \} \) has a pure strategy Nash equilibrium.

**Proof.** Let \( BR_i : S_{-i} \to S_i \) be the best response correspondences of player \( i \) to the pure strategies \( s_{-i} \) of all other players. Define, in line with (2.25), the best response correspondence \( BR : S \to S \) as

\[ s \in BR(s) = \left[ BR_i(s_{-i}) \right]_{i=1}^I \]

\( S \) is non empty, compact and convex as the Cartesian product of the \( S_i \)’s.

On the other hand, quasi concavity of \( u_i(s_i, s_{-i}) \) in \( s_i \) implies that for any \( s_i', s_i'' \in S_i \)

\[ u_i\left( \lambda s_i' + (1-\lambda) s_i'' \right) \geq u_i\left( t_i, s_{-i} \right) \]

for all \( t_i \in S_i \)

Hence, \( \left( \lambda s_i' + (1-\lambda) s_i'' \right) \in BR_i(s_{-i}) \), thus showing that \( BR_i(\sigma_{-i}) \) is convex-valued.

The generalization of Kakutani’s theorem can then be used to prove the existence of a Nash equilibrium in pure strategies in continuous games.

**Example.** Consider Cournot’s quantity strategy duopoly game. Let \( q_i \in Q_i = [0, \bar{q}] \in \mathbb{R} \) \( i = 1, 2 \) be the strategy set of each duopolist;\textsuperscript{27} \( p(q) = a - bq \) be the inverse demand function with \( q = q_1 + q_2 \); and \( c_i q_i \) the cost function of firm \( i \). The profits of firm 1 are

\[ \Pi_1(q_1, q_2) = \left[ (a-c_1) - b(q_1 + q_2) \right] q_1 \]

\textsuperscript{26} The standard reference is to three papers, all published in 1952, by Debreu, Glicksberg and Fan. In Glicksberg and Fan the question of existence of a fixed point is stated for a correspondence defined on a convex, compact subset of a locally convex Hausdorff linear space; in Debreu the question is stated in terms of a contractible polyhedron. In J. Duggan’s Note “A slight twist on Glicksberg Theorem” (2002), the question is analyzed with reference to subsets of a finite dimensional Euclidean space. The same approach is followed by A. Ozdaglar, “Lecture Note 5: Existence of Nash Equilibrium”, (Lecture notes, February 21, 2008). The presentation adopted here follows Ozdaglar’s very neat sketch of the proof, also in consideration of the fact that metric spaces are of basic interest to economists.

\textsuperscript{27} We could set \( \bar{q} = q^M \) where \( q^M \) is the output that maximizes monopoly profits.
For given $q_2$, the second derivative of the profit function with respect to $q_1$ is

$$\frac{\partial^2 \Pi_1(q_1, q_2)}{\partial q_1^2} = -b.$$ 

$\Pi_1(q_1, q_2)$ is therefore concave and, a fortiori, quasi concave. The conditions for the existence of a pure strategy Nash equilibrium are thus met. Crucial to this end is the assumption that the profit function is concave which in the example is due to the assumption that the inverse demand function is concave and the cost function linear.
References


